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Abbacus geometry – a tradition without pretensions and without future

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Fractures

The topic for this last lecture is Italian abbacus geometry.

This topic nobody (myself included) has ever taken up specifically, for the valid reason that is in not very interesting if looked at solely as “mathematics”. It is, as my heading says, “without pretensions and without future”.

So, it has to be looked at for different reasons.

Yet it remains “mathematics”, and mathematics must remain within the perspective.

Abbacus geometry is “nothing but a subordinate topic within practical geometry”, one may say.

“Practical geometry”, however, are many things, which have not much more in common than

- *not* being founded upon Euclidean proofs;
- or *not* being taught by means of these;
- or *not* being cultivated as dealing with intelligible mathematical entities
- or *not* ... .

Speaking of “practical geometry” is like taking one rose out of a bouquet of flowers and speaking of the others simply as “not-that-rose”.

It belongs to the same tribe as “non-yellow colours”, giving only an illusion of insight in the specificity of the constituents.

Or, more provocatively, as “non-Western mathematics”, as if the only important thing to say about Chinese, Sanskrit or Aztek mathematics was that they are not “Western”.

“Western”, by the way, also being a highly misleading term, intimating that it should refer to one thing. I hope my previous lectures have made it clear that it is not, and in particular not the “thing” with which current ideology identifies it.

With the arrival of printing and the increase of literacy, the boundaries between the various “practical” geometries that thrived in Europe, and between these and “not-practical” (i.e., Euclid-emulating) geometry become porous.

That is evident in the writings of Italian 16th-century architects – but also though more modestly Mathes Roriczer’s *Geometria deutsch* from 1487–88.

This is not my topic, so I shall not go into details. Others have done so.

Surviving sources from earlier epochs which are now and and were when they originated classified as “practical” geometries mostly reflect the work of those who calculated on the basis on measurements made by others;

we may speak of “scribal practical geometry”.

The methods used by master builders for their constructions were handed down within a master-apprentice network.

Rarely, but only rarely and indirectly they made their way into writing.

We may therefore disregard genuine construction and surveying, and concentrate on the scribal variant;

but even then we are forced to recognize divisions already within a restricted area like Latin Western Europe during the High and Late Middle Ages.

Whoever has looked at Hugh of Saint Victor's *Practica geometriae* from the 1120s will have noticed the division into *altimetria*, *planimetria* and *cosmimetria*.

These terms cast long shadows in the Latin tradition – they are still obliquely reflected in Christian Wolff's 18th-century mathematical encyclopedias.

But they leave no trace in the abbacus geometries.

So, if we want to speak of "a tradition", we cannot go beyond abbacus geometry:

> it is wholly separate from what was inaugurated by Hugh, and also from that post-Boethian geometry from the Latin Early and Central Middle Ages, on which Hugh's work can be considered the crown.

Since I shall need to refer to them repeatedly in the following, I shall say a few words about this Latin tradition and its components.

In Roman Antiquity, several agrimensors included some measuring geometry (without any kinds of proofs) in writings with a broader scope.

Columella did so too in his first-century book *On agriculture*.

Around 500 AD, Boethius tried to save the basics of ancient philosophical and mathematical heritage, translating among other things an epitome of Euclid's *Elements*.

Hardly, as sometimes asserted, the original work or its first books.

As the need for instruction of administrators materialized in the "Carolingian Renaissance" after 780, surviving agrimensor- and Boethius-manuscripts came to provide the basis for geometry teaching at cathedral schools.

Its writings are thus "post-Boethian" and "post-agrimensor".

Abbacus geometry – a tradition takes form

In the beginning abbacus geometry did not look much as “a tradition”.

How could it?

New traditions are not born like the Greek goddess Athena, fully armed when making the leap into the world.

That holds for Greek deductive geometry, as I argued in my third lecture, but also for much more modest undertakings with much less impact on the world.

The first abbacus masters who collected matters geometric were forced to draw eclectically on what they found available and adequate

– each on his own and borrowing from his own background.

Most likely, the genuine beginnings escape us.

However, the *earliest surviving* abbacus writings which deal with geometry still reflect these beginnings – just as the earliest abbacus arithmetical writings reflect the multifarious beginnings of *that* tradition.

Neither type was created as a simplification of Leonardo Fibonacci's great Latin works, however much this is the standard fable.

Even though we do not know the first beginnings, it is not difficult to go behind them and trace much of the material to earlier mathematical cultures.

I shall offer some scattered observations on this kind of historical background where it can serve to create perspective; but nothing systematic.

First some words about the Italian abbacus school, the soil from which abbacus arithmetic and abbacus geometry grew.

As I have told in earlier lectures, it thrived from the second half of the 13th until the first half of the 16th century between Genoa, Milan and Venice to the north and Umbria to the south.

It mainly taught basic numeracy and commercial arithmetic, but also some geometry.

Everything was done by means of Hindu-Arabic numerals – *abbaco* is only etymologically related to *abacus*.

The earliest abbacus text that has come down to us is probably the *Columbia algorism*.

We do not possess the original, only a 14th-century copy, edited by Kurt Vogel in 1977. I shall refer to his section numbering.

Because of mistaken identification of certain coins appearing in a coin list, Vogel dated the treatise to the second half of the 14th century, while observing that numerals are written as they would have been in the 13th century.

Better informed by more recent numismatic research, Lucia Travaini has since then been able to date the coin list to "after 1278 and before 1284".

That does not necessarily mean that the rest of the treatise was written during these years – coin lists were often borrowed from earlier writings.

Yet Vogel's observation of the shape of numerals suggests a 13th-century date for the original. The genre did not invite the creation of fake evidence of high age.

Geometry is mainly found in a closed group of problems (except for the insertion of the coin list between the second-last and the last problem).

Section #125 explains what a square root is, and that square roots can be found in and solve all geometric problems.

This is a slight overstatement, roots do not enter in all the geometric problems that follow;

but it reflects the fact that square roots are always introduced in the geometry chapters of abbacus books.

Section #126 explains how to approximate roots of non-square numbers.

The method it teaches was to become the standard of abbacus geometries (called the "closest approximation"), namely

$$\sqrt{n^2+d} \approx n+\frac{d}{2n} ,$$

exemplified by $\sqrt{10} \approx 3+{}^{1}/_{6}$.

In principle, the method could be used from below, as here, or from above, for instance $\sqrt{8} \approx 3-{}^{1}/_{6} = 2{}^{5}/_{6}$,

but here as in almost all later abbacus geometries only approximation from below is taught (which would give $\sqrt{8} = 2+{}^{4}/_{4} = 3$).

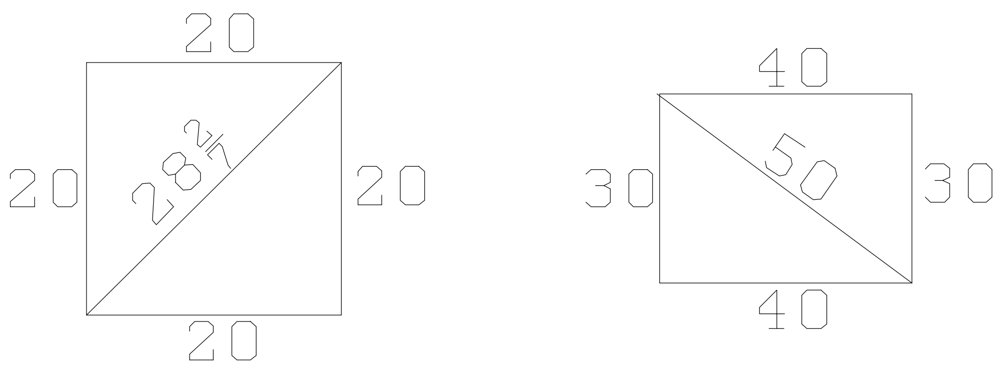
It is claimed that better cannot be done; even this corresponds to most later abbacus geometries.

Geometry proper starts with #127, which deals with a square piece of land with side 20 (no unit is indicated).

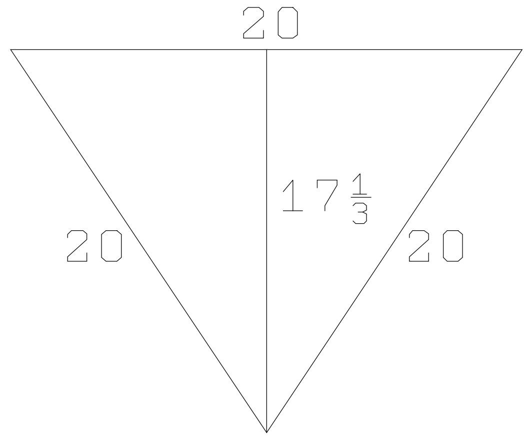
From what I shall call the “Pythagorean Rule” it follows that the diagonal – spoken of as the distance “from corner to corner” – is $\sqrt{800}$, approximated as $28^2/_7$.

“Pythagorean rule”, not “theorem”, since neither explicit enunciation nor proof is offered.

#128 treats of a piece of land which on one face is 30, and in length 40 (thus a rectangle); the diagonal can this time be found exactly, as 50.



#129 has as its object “a field made in the way of a *schudo*” (a “shield”, but here as mostly elsewhere much too large to be a real shield – the side is *ca* 10 metres).



In the present case, and mostly in abbacus geometries, the term refers to an equilateral triangle;

but at times to an isosceles or scalene triangle; when so, both or all three sides are given, eliminating ambiguity.

This term is used in all the early abbacus geometries I present here, showing that they share one strand of their background in spite of the differences they exhibit;

diagrams invariably correspond to the term, showing what we would speak of as the “base” on top.

The term as well as the diagram show that neither Fibonacci's *Pratica geometrie* nor any known Latin geometry have provided the inspiration.

The height is found as $\sqrt{20^2-10^2} = \sqrt{300}$, approximated as $17\,^1/_3$ (probably a secondary rounding, the first approximation being the usual $17\,^{11}/_{34}$).

If the diagram had shown that half of the upper side is 10, this could have been seen as a hint of the underlying argument. But it is not, nor is the halving of 20 made explicit. Arguments are generally absent, in the present and in most other abbacus geometries.

In abbacus *arithmetic,* in contrast, intuitive arguments are regularly given where they can be judged to be within the reach of the reader (and that of the writer, for sure). Genuine proofs are absent from both genres.

#130 is the first question about a circle.

In their treatment of the circle, abbacus geometries fall in two classes:

some take the perimeter (“as much as it is around”, and similarly) as the basic parameter,

in others (among which the *Columbia algorism*) the fundamental parameter is the diameter (*il mezzo*, “the middle”, and similarly, often written $^1/_2$).

All, however, take the perimeter to be $3^1/_7$ times as much as the diameter, without any hint that this is an approximation.

#131 is the inversion of #127, stating that the diagonal of a square is 10, and finding the side as $\sqrt{(10^2 \div 2)}$, approximated in the usual way as $7^1/_{14}$.

#132 is a similar inversion of #130. It asks for the diameter of a circle with perimeter 20.

#133 and #134 deal with a rope connecting the top of a tower and a point at a certain distance from the tower

> in most abbacus geometries this point is the other side of a moat, here it is a fountain.

In #133 the length of the rope is 50 cubits, and the distance is 30. The height of the tower follows from the Pythagorean Rule.

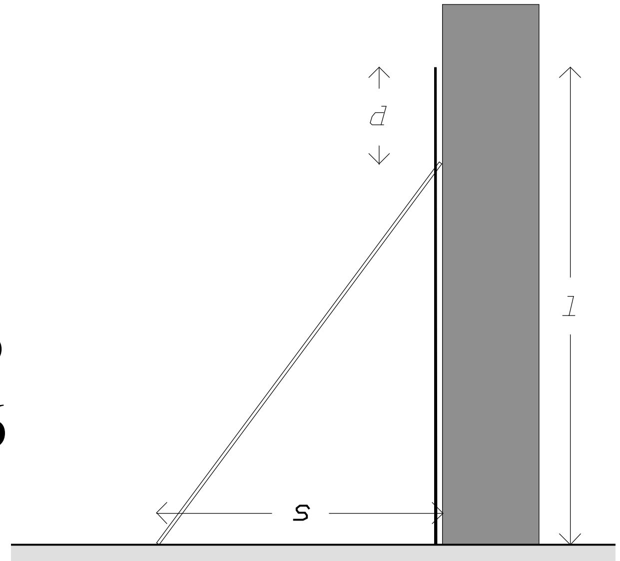
In #134, the length of the rope is determined from the other two parameters.

These problems will conventionally be classified as “recreational”:

they deal with what seems to be a possible real-world situation (duly simplified, a real rope would hang and not be straight); but they would not be encountered in the eventual professional practice of the learner.

The name “recreational” corresponds to the function of such problems in modern times: originally, they rather served to test and display the ability of the practitioner.

These two problems ask for nothing but a modicum of ability. #135, also "recreational", is only slightly more difficult.



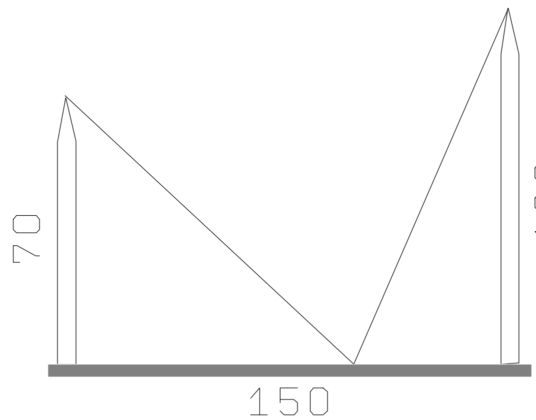
A ladder of l = length 10 cubits stands against a wall, also 10 cubits high. Then the foot of the ladder slides out $s = 6$ cubits, and it is asked how much (d) the top of the ladder moves down.

Once more, the Pythagorean Rule solves the problem.

The problem is of venerable age. It is first found in an Old Babylonian text from *ca* 1600 BCE.

Together with a more sophisticated version where the two slidings (s and d) are given it turns up again in Seleucid and Demotic texts (*ca* third century BCE).

#136 is still “recreational”, and more outspokenly so.



It speaks about two towers of heights 100 cubits and 70 cubits respectively, and distant 150 cubits from each other.

Somewhere between them sits a duck, and on top of each tower sits a falcon. The two falcons leave at the same time and reach the duck at the same time (flying with the same speed).

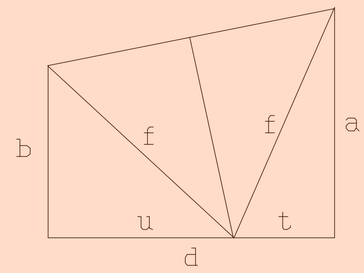
Even this problem is of old age (though not comparably old) and widely diffused. Mostly, the birds are doves, and their aim is a fountain from which they drink.

The earliest appearance I know of is in Mahāvīrā’s *Gaṇita-sāra-saṅgraha* from *ca* 850 CE, where the principle of the solution is explained.

The distance between the duck and the highest tower is determined as $(150^2+70^2-100^2)\div2\div10$. This is indeed true, and comes from double application of the Pythagorean rule.

I shall skip the proof but may return to it if anybody asks.

According to the Pythagorean Rule,



$$f^2 = a^2+t^2 = b^2+u^2 \ ,$$

whence

$$a^2-b^2 = u^2-t^2 = (u+t)\cdot(u-t) = d\cdot(u-t) \ ,$$

and thus

$$\frac{u-t}{2} = \frac{a^2-b^2}{2d}$$

and thereby

$$t = \frac{u+t}{2}-\frac{u-t}{2} = \frac{d}{2}-\frac{a^2-b^2}{2d} = \frac{d^2-a^2+b^2}{2d} \ .$$

The second-last step corresponds to what is hinted at by Māhāvirā. The final transformation, corresponding to the rule given in the *Columbia algorism*, was certainly not derived by the algebraic calculation I offer here. It may perhaps have been found by some analogous way of reasoning – but in that case certainly not by the *Columbia* compiler.

#137–139 return to what can be used in professional practice (several abbacus masters are known also to have been active in urban surveying).

- #137 finds the area of a square area with side 60 cubits
- #138 that of a rectangular field with length 80 cubits and width 40 cubits
- #139 that of an irregular field with “lengths” 30 and 50 and “widths” 20 and 40.

The latter area is calculated by means of the “surveyors formula”
average length × average width.

In real surveying, this approximate formula was probably only used for near-rectangular shapes, where it is quite satisfactory. An area as irregular as the present one would almost certainly have been split into triangles.

#140, coming after the coin list, looks like an afterthought.

It is of type ladder-against-wall, but makes use of the unlikely dress of a tree standing along the wall (both 20 cubits tall) and then moving 10 cubits away. Mathematically there is nothing new.

The geometric material contained in the *Columbia algorism* recurs in all later abbacus geometries.

However, area calculation, secondary in the *Columbia algorism*, was going to be prominent, and a number of further problem types would turn up – not least stereometric problems, very often badly understood.

The *Primo amastramento de l'arte de la geometria*, "First Teaching of the Art of Geometry", is probably slightly but not much later than the *Columbia algorism*, and also known from a 14th-century copy only.

It is the companion piece to a *Livero de l'abbecho*,

This latter compilation pretends to be made *secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa*, "according to the opinion of Leonardo Fibonacci".

Actually, the *Livero* consists of two separate though interwoven components

- one a basic abbacus treatise corresponding to the school curriculum,
- the other badly understood extracts from the *Liber abbaci*.

This Umbrian treatise has been dated to the years 1288–90 on the basis on loan contracts which it contains.

Closer analysis shows, however, that these are borrowed from an earlier work, and therefore only provide a *post quem* date;

most likely, it is from the beginning of the 14th century.

The *Primo amastramento* is an independent treatise, but evidently meant to accompany the *Livero*.

The two are similar, but there are some stylistic discrepancies, suggesting

- that they are either written by different but associated compilers
- or that the compiler developed his style in the process.

As in the *Livero* there are some borrowings from Fibonacci (still from the *Liber abbaci*, not from the *Pratica geometrie*); they are quite modest, however.

The *Primo amastramento* differs from the geometry of the *Columbia algorism* in several respects.

Some of these makes it agree with later abbacus geometry, while others disagree.

The *Primo amastramento* approximates irrational square roots according to the same formula as the *Columbia algorism*, but when it fits from above;

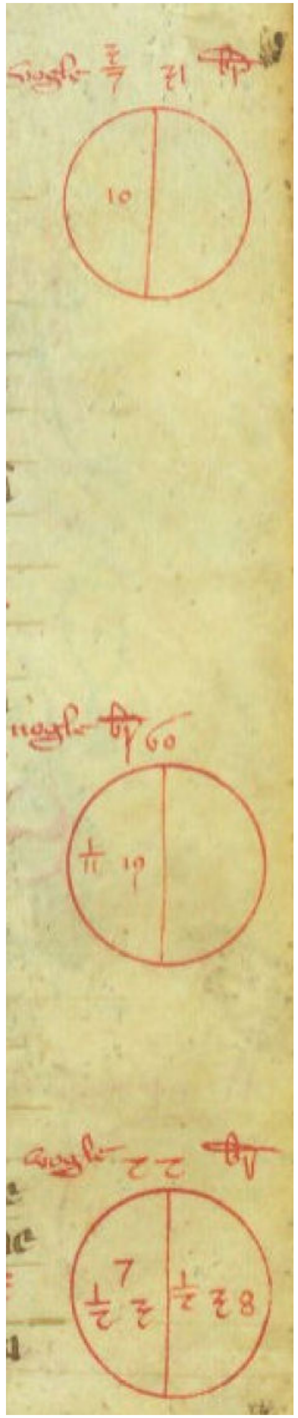
this is certainly better mathematically, but on this account later abbacus geometry follows the *Columbia algorism*.

A second difference is that area measurement turns up in full force.

A third general difference is presented by the approach to the circle.

In the *Prima amastramento*, the fundamental parameter is the perimeter, the diameter being found via division by $3\frac{1}{7}$.

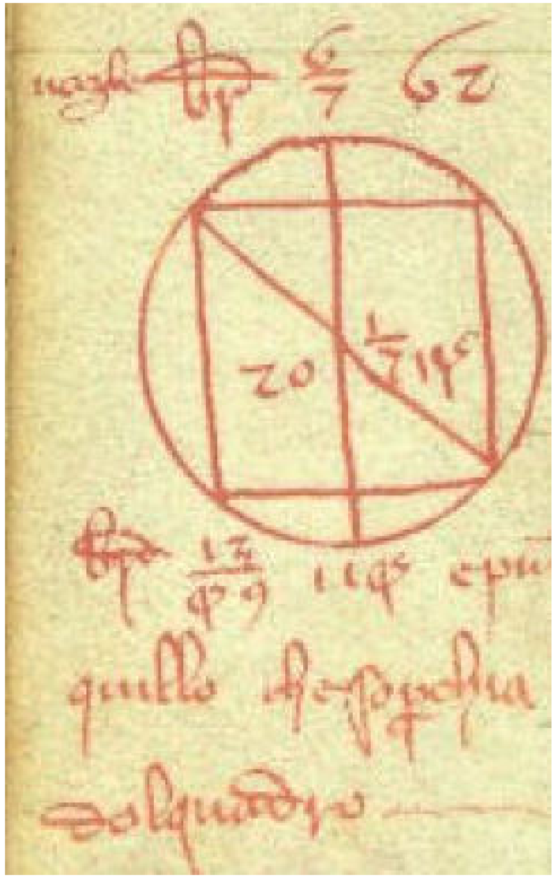
The area (not dealt with in the *Columbia algorism*) is found either as ($\frac{1}{2}$ perimeter) × ($\frac{1}{2}$ diameter) or as $\frac{1}{4}$(perimeter × diameter).



To this comes a wider range of problems.

Some deal with the inscription of equilateral or isosceles triangles (even here *schudi*), squares and circles in each other, mostly but not always solved correctly.

Some solutions are not only wrong but evidently corrupt – the compiler apparently did not understood too well the material he borrowed.

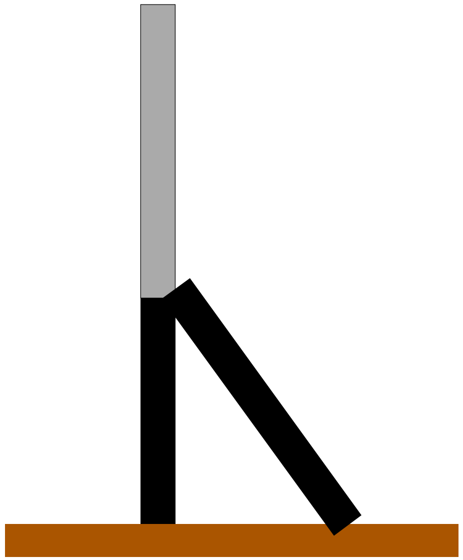


Many problems involving right triangles (dressed in various ways) seem to be underdetermined; in all of these the triangle is silently supposed to be proportional to the 3–4–5 triangle.

One of them emulates the two-tower problem, but it is given that both birds fly 100 cubits, after which the 3–4–5-model is used to postulate fitting heights of the towers.

Another one is of type ladder-against-wall; it thereby gets an equally cheap solution.

Other problems deal, without this cheap trick, with a kindred and equally widespread problem type:



A tree of known length is broken, and either the height where it breaks or the distance where the top falls is given.

The solution given is correct, but it seems clear that the compiler does not know the underlying simple reasoning.

Several problems ask for the addition or subdivision of circles.

They are formulated as if dealing with real pieces of land;

what is found, for instance, when a circle is divided into four is the diameter of circles whose areas are $^1/_4$ of the original area.

Evidently, the writer is a mathematics teacher who constructs problems that look “practical” but have nothing to do with practice.

The practice he knows best is the *practice of teaching*. A familiar phenomenon

Others ask simple questions about concentric circles.

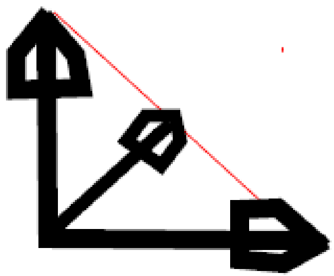
One problem deals with a tree of height 20 cubits which is felled, falling 1 cubit for each cut.

The number of cuts is determined (sensibly) from the length of the arc described by the top of the tree.

In a seemingly analogous problem elsewhere, a tree of length 40 cubits is raised 1 cubit per day.

Here movement is (tacitly) measured along the straight line from the original to the final position, and the duration of the process is claimed to be $\sqrt{(s \cdot 40^2)}$ days.

It is not made clear whether the top of the tree is believed to follow this line, or only the motion of the point of section is meant (but the latter concept is probably too intricate for the compiler).



Some problems are only geometric “by association”, that is, because they are dressed as dealing for instance with a tree.

The same types are sometimes found within geometry sections of later abbacus books, sometimes outside.

Their counterparts in the *Liber abbaci*, when they have any, are found outside the geometry section.

There is no reason to discuss them further.

In contrast to the *Columbia algorism*, the *Prima amastramento* deals with stereometric problems.

Some are quite simple – thus, the determination of the contents of a circular well with depth 20 cubits and circumference 22 cubits.

Reasonably, the area of the cross-section is said to be the volume for depth 1 cubit.

Accordingly (we may say), the unit for volume is the same as for areas, "square cubits" – this corresponds to what is done in all other abbacus geometries.

Others speak about a well or a circular fish pond of given dimensions into which some other body falls

(a cylindrical column, a cone, a double cone or a parallelepiped, also with known dimensions).

If the well is full, it is asked how much water flows out; if not, how much the water level is raised.

In some of these problems the contents of the well (in these cases specified to be a cistern) is stated in *bariglie* (“barrels”, a hollow measure), which leads to the need to determine the ratio between “square” cubits and *bariglie*.

This more complex variant is found both in the *Liber mahameleth* (Latin *ca* 1160, Arabic original perhaps *ca* 1130) and the *Liber abbaci*, both of which also speak of a cistern.

After a section with number problems follows on fol. 175^r more geometry, *regole ch'è de misurare terre*, "rules for measuring land".

Here we find at first a description of Assisi metrology. A similar list but of *Pisan* metrology is found in Fibonacci's *Pratica geometrie*.

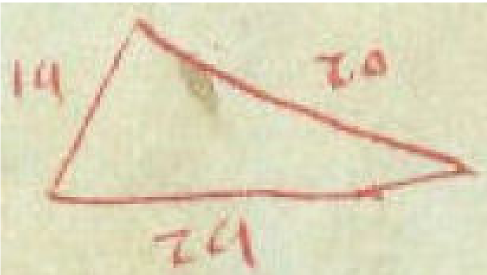
> The organization is completely different; there is no reason to suspect even slight inspiration – both texts respond to a local need.

Afterwards come areas of rectangles with complex measures, and then that of an almost-rectangular house determined by means of the "surveyor's formula".

This section was clearly meant to teach that kind of geometry which abbacus masters would need in urban surveying.

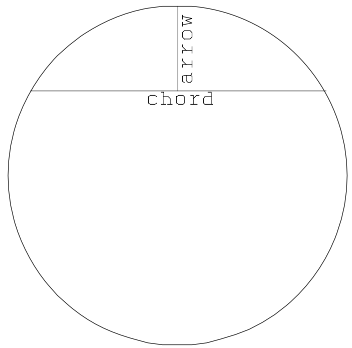
In the very end (fols 177ʳ–178ᵛ) come three problems which appear to have a different provenience.

One finds the area of a triangle with sides 25, 20 and 15 cubits – that is, of 3–4–5-proportions – but by means of the height on the hypotenuse.



The calculations are correct, but the compiler does not seem to understand. Originally, this calculation may have been meant to show that this alternative way gives the correct solution.

The second deals with a column which in part is under ground.



Actually, this dress is nothing but a pretext for determining the dimensions of a circle from the chord and the arrow of a segment – here done wrongly.

The third compares the weights of two wax spheres with diameters 3 and 5 (no unit given). It only calculates 3·3·3 and 5·5·5 and then stops, as if the compiler did not understand the purpose of this calculation.

We shall encounter two more problems dealing with wax spheres, both erring.

All in all, the dress is rather rare. The few occurrences may have been imported laterally from the Latin tradition on distinct occasions, by incompetent writers who tried their own hand.

Even the *Liber habaci* is known from a 14th-century copy only

The original can be dated by calendarian material to *ca* 1309,

and the appearance in this material of saints' days that were important in Provence but not in Italy testifies to the place where it was written.

The ascription to Paolo Gherardi (on whom later) by an 18th-century librarian can be safely disregarded.

The *Liber habaci* is unique among abbacus books by not using Hindu-Arabic but Roman numerals throughout.

Apart from not teaching computation with Hindu-Arabic numerals it is a full abbacus commercial arithmetic,

dealing with computation with mixed numbers, with the Rule of Three, and with interest, exchange of monies, metrological shortcuts, partnership, barter, local metrologies.

It also contains a coin list, and calendar-reckoning with astrology;

and then, in the end of a section on the arithmetic of fractions *a presentation of roots.*

and between the shortcuts and partnership *a geometry.*

The presentation of roots outside the geometry is unique in abbacus books.

So is also the way irrational square roots are approximated:

$\sqrt{2} = 1\,^{3}/_{7}$, a little less	$\sqrt{12} = 3\,^{1}/_{2}$, a little less
$\sqrt{3} = 1\,^{3}/_{4}$, a little less	...
$\sqrt{7} = 2\,^{2}/_{3}$, a little less	$\sqrt{40} = 6\,^{1}/_{3}$, a little less
$\sqrt{10} = 3\,^{1}/_{6}$, a little less	$\sqrt{50} = 7\,^{1}/_{14}$

$\sqrt{10}$, $\sqrt{40}$ and $\sqrt{50}$ *could* have been found as the usual "closest approximation", but the rest of the table shows that the underlying idea (not stated in the text, and probably unknown to the compiler) is

$$\sqrt{2} = \sqrt{(2\cdot 49)}/7 \approx \sqrt{100}/7 = 10/7$$
$$\sqrt{3} = \sqrt{(3\cdot 16)}/4 \approx \sqrt{49}/4 = 7/4$$
$$\sqrt{7} = \sqrt{(7\cdot 9)}/3 \approx \sqrt{64}/3 = 8/3$$
$$\sqrt{10} = \sqrt{(10\cdot 36)}/6 \approx \sqrt{361}/6 = 19/6$$

etc.

Only $\sqrt{50}$, not said to be an approximation, comes from age-old tradition.
The method asks for more ingenuity than the standard procedure, which may be a reason it left no later traces:

In order to find $\sqrt{q}$, one has to find a number n such that qn^2 is close to a square number r^2 (a Pell equation if $r^2 - qn^2 = 1$).

For this, no automatic routine was at hand, as it is for the usual "closest" approximation from below.

Later abbacus writers on geometry obviously preferred a routine that asked for no thinking – not even about whether to approach from below or from above.

The geometry starts by setting out Florentine area metrology, and then goes on with the area of rectangles with somewhat intricate measures and metrological conversions.

This is similar to the end of the *Primo amastramento*, we remember.

The section on the circle (*tondo a sexta*, “compass-made round”) takes the perimeter as the basic parameter, and as usually uses $3\,{}^{1}\!/_{7}$ for π.

The diameter is found by division, and the area as ${}^{1}\!/_{4}\cdot$(perimeter$\times$diameter).

Afterwards correct solutions are given to opaquely formulated requests to inscribe a maximal square and (equilateral) *schudo*, “shield”, within a circle,

and then two about a segment with arrow a and chord c.

The first is dressed as a wheel partly borrowed in the ground (thus an analogue of the segment problem in the *Primo amastramento*), and solved as for instance by Abū Bakr and Savasorda.

The second, dealing really with a bow with arrow and chord, claims that the area of the segment is $^{11}/_{14}\, c \cdot a$

This is true for the semicircle, for which it is given by Columella and in the post-agrimensor *Geometria incerti auctoris*.

For the present segment, it is false, as Columella as well as the "uncertain author" know.

The end of the circle section asks for a circle being transformed into a *schudo* of the same area. No answer is given.

The area of the *schudo* is indeed only dealt with afterwards, first the equilateral triangle, done as in the other treatises we have looked at,

then a scalene triangle with sides 7, 8 and 9 cubits, whose area is found by means of “Heron’s formula”.

Next the text returns to squares and rectangles, asking for areas and diagonals (or for determinatoion of a square side from the diagonal),

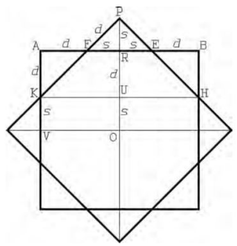
and then for transformation and inscriptions – very similar to what we have seen elsewhere.

Noteworthy is only a problem about a square, in which the difference δ between the diagonal *d* and the side *s* is given.

The solution, offered without explanation, is $s = \delta+\sqrt{}(2\delta^2)$, $d = 2\delta+\sqrt{}(2\delta^2$.

This is obviously related to the side-and-diagonal-rule, known since classical Antiquity. The rule states that if δ and σ are side and diagonal of a square, then $s = \delta+\sigma$ and $d = \delta+2\sigma$ are so too.

I have not noticed anything similar in later abbacus geometry, but a related construction of the regular octagon is found in Roriczer's *Geometria deutsch*.



The end of the section deals with box-shaped volumes of piled box-shaped stone slabs (walls made by bricks; quite simple).

Then begins a section *Della torre e del poçço e del vivaio*, “about the tower and wells and the fish pond”.

It is kept together only by these vertically extended objects (all recreational favourites, but here also a fortress is included).

No mathematical principle is shared:

Some deal with concentric circles with given distances;

some (either speaking of a tower and a rope or of a rod leaning against a wall) with application of the Pythagorean Rule,

some (speaking of division of circular objects) with simple area calculation (not taking physical constraints in consideration).

One problem from the section is the two-tower problem, solved by means of a rule that only happens to be valid because of the specific parameters.

- Namely because $2ab–2b^2 = d^2–(a^2–b^2)$, with the letters I used before.
- The rule has probably been constructed by somebody who knew the configuration and then played around with the numbers until they gave the intended result.

Another problem deals with a tree which every day falls 1 cubit, here tacitly measured along the straight line from initial to final position (another application of the Pythagorean Rule).

Some deal with stereometry proper, for instance construction of houses or platforms by bricklaying or the familiar stone falling into a well or a fish-pond.

One type (familiar from the *Liber abbaci* as well as later abbacus writings) compares the grain contained in two cubic chests of known dimensions – here with sides 4 and 2 cubits.

Often but not here a possible fraud is involved – will it be adequate to render 2 times the contents of the smaller chest if one has borrowed what was in the larger one?

Later *torre e poççi e fosso* return, “towers and wells and moat”. Nothing of particular interest is to be found there.

10 minutes' break

These three works belong to the prehistory of abbacus geometry.

With their eclectic and disparate borrowings from earlier traditions they are too different to be considered already part of a tradition. They reflect a need, and they are (as yet tentative) responses to that need.

There is already some uniformity of terminology, not least the singular use of *schudo* for a triangle which by default is equilateral but can also be specified to have sides that are not equal.

As said, they must thus have some measure of shared backgrund.

Other terms, however, are clearly explanations in everyday language, and not yet fully standardized.

Though apparently older than the (archaic or archaizing) *Liber habaci* by a few years, the geometry chapter in Jacopo da Firenze's *Tractatus algorismi* shows us the beginning of the mature tradition.

According to its colophon, the *Tractatus* was written in Montpellier in 1307.

Three copies survive, likely to be from the early to the mid-15th century.
All share a geometry section apart from minor differences.

At first the circle is dealt with. The fundamental parameter is the perimeter, and as always π is taken to be $3^1/_7$.

This is even stated as a kind of axiom,

> if you should want to know for which cause you divide and multiply by 3 and $^1/_7$, then I say to you that the reason is that every round of whatever measure it might be is around 3 times and $^1/_7$ as much as is its diameter, that is, the straight in middle.

The area is found as $^1/_4\cdot$(perimeter × diameter).

Applications of the Pythagorean Rule follow, first to the hypotenuse of a right-triangular field, next to the diagonal of a field with side 10 cubits.

Here, 100 is approximated as $14^1/_7$, which is said to be "the closest, because precisely it cannot be found".

After that we find one of the usual non-geometric interlopers (and probably the most common type), a problem of type *leo in puteo* about a snake climbing a tower.

– The problem type gets its conventional name from a problem in the *Liber abbaci*, in which a lion climbs out from a pit.
– Here instead, as more often, a snake climbs a tower 30 cubits high, by day scending $^1/_3$ of a cubit, by night sliding down $^1/_4$.
– Both forget that once the animal has reached the goal by day it does not matter for the answer what happens afterwards. Thereby, the problem is reduced to fraction arithmetic.

Then Jacopo returns to geometry proper,

first asking for and finding the area of a rectangle,

then presenting two tower-moat-rope problems (with the invariable 40–30–50 parameters),

and then (with a cross-reference “I have also said it to you above”) a question for the diameter of a circle with perimeter 100 cubits.

A section about square roots follows.

First the explanation of the concept is given together with a list of the square roots of the perfect squares until 121, with indication that one may continue “with every other number that is multiplied in itself”.

Then the rule for finding the “closest root” by means of the standard approximation from below, meticulously explained for $\sqrt{10}$, and then illustrated by the further examples $\sqrt{67}$ and $\sqrt{82}$.

After this we first find a typical instance of schoolmaster's practice-free "practical" geometry.

On an area 567 cubits long and 31 cubits broad houses 11 × 7 cubits are to be built.

The number of houses that can be built is asked for, and calculated without consideration of the fact that with their given dimensions the houses cannot fit – the answer given is 228 houses and $^{3}/_{11}$ of a house.

The problem type (and the disrespect for the implications of the dress) is borrowed from the Latin tradition.

Here, trapezoidal, triangular and circular fields are encountered, even further removed from genuine practice.

The first indication that Jacopo had no spatial intuition is a problem speaks about a well and a stone falling into it – here a column.

It measures 2×2×50 cubits and is full of water, while the column is 1×1×25 cubits. It is asked how much water flows out.

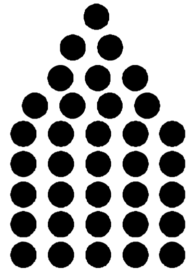
As we see, this is the simple version which we encountered in *Primo amastramento*, but here the calculations are wrong,

> probably caused by some interaction with the complicated version in which the ratio between volume and hollow measure has to be determined.

Jacopo finds, correctly, that the volume of the column is $^{25}/_{200} = ^{1}/_{8}$ of that of the water, and concludes that 8 “square” cubits flow out.

The appearance of the mature tradition, as we see, was not accompanied by better mathematical understanding, nor did it produce it.

That is also illustrated by the next section. It asks for the area of a regular pentagon with side 8 cubits, and gives this rule:



> Multiply one of the faces by itself, that is, 8 times 8, which makes 64. Now multiply by the other three faces, 3 times 64, it makes 192. Remove from it one of the faces, that is, 8. 184 is left, and it is done.

This claim is evidently inspired by the formula for the nth pentagonal *number*, $\frac{1}{2}\cdot(3n^2-n)$, known through the ancient Roman agrimensor- and later medieval post-agrimensor tradition, where this mistaken use of polygonal numbers as area measures is familiar.

That the factor $\frac{1}{2}$ is omitted is exactly that – an omission;

but the reference to $3n$ as "the other three faces" evidently results from somebody (Jacopo's source rather than Jacopo himself) trying to connect the formula to the actual geometric configuration.

Stereometry returns with a pavilion supposed to be conical.

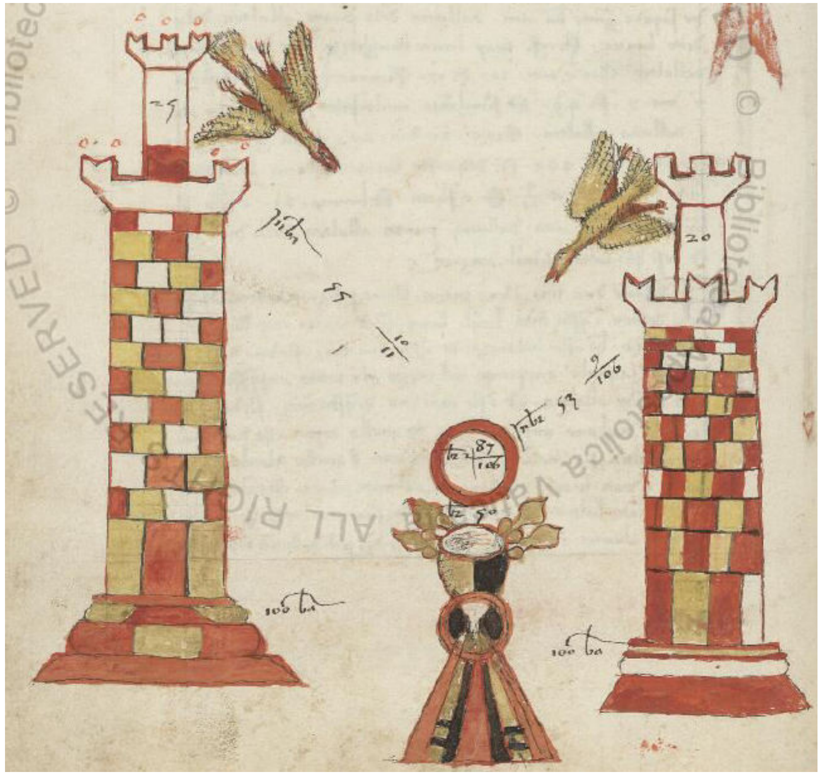
It has a mid-pole high 40 cubits, and the distance from the peak of the pole to the border of the cloth is 50 cubits.

The half-diameter and hence the diameter (60 cubits) are found correctly by means of the Pythagorean Rule, from which follows, still correctly, the area that is occupied.

The area of the cloth, however, is found as $(\frac{1}{2}\cdot 60)\cdot 50$ – the area of a triangle with height equal to the peak-to-border distance and base equal to the *diameter* of the circle on the ground instead of the *perimeter*.

This can hardly be explained as a mistaken copying from a correct source.

Second-last in the geometry chapter we find a cheap version of the two-tower problem.



The goblet from which two birds want to drink is placed at the mid-point between the two towers, and the question is, how much earlier one bird arrives than the other.

This, of course, could not be answered in an epoch where the concept of quantified velocity had not yet been created;

reasonably, instead, Jacopo calculates the difference between the two distances (once again using the Pythagorean Rule).

Already in the preceding chapter (containing mixed problems), there is some geometry (other abbacus authors would indeed put the problems in question into their geometry chapters).

Noteworthy is this:

> A hall, or indeed *piazza*, is 120 cubits long, and 36 cubits broad, neither more nor less. And I want to flag it with flags or slabs that are all of one and the same magnitude. And each slab is $^1/_2$ cubit long and $^1/_4$ broad. I want to know how many slabs are required to flag the said hall.

Noteworthy not because of its mathematics, which is unproblematic, but because of its prehistory and its further career.

It comes from the Latin post-agrimensor tradition, and it became a stock problem type in abbacus geometries.

A similar problem asks for the number of bricks of given dimension that go into a wall of given dimensions. Even that became a stock problem.

Paolo Gherardi

A *Libro di ragioni* contained in a Florentine manuscript states in its colophon to be written in Montpellier in 1327 *secondo le regole e'l corso dell'ambaco facte per Paulo Gherardi di Fierentie*, "according to the abbacus rules and course held by Paolo Gherardi of Florence".

This probably means that the text was written down by an assistant or mature participant in the course.

In any case the colophon tells us that Gherardi was an abacus teacher. With this caveat we may speak of the text as "Gherardi's *Libro*".

Gino Arrighi made an edition in 1987, which I use.

The treatise is not divided into chapters, but the single problems carry a header stating their topic.

What concerns us here are those dealing with *missure* and *giomatria* (and a few about *chadrare* and some other specific characterizations). There is no systematic progression.

Quite a few problems are corrupt.

Some of the mistakes may have arisen in copying.

Others, however, certainly come from the original compilation.

Area measures are important. When correct they are calculated as we have seen before (there is indeed not much choice).

For the circle the diameter and not the perimeter mostly serves as basic parameter.

Often such problems are embedded in composite questions, for instance comparison of two areas (at times simple volumes), or in transformation of one shape into another one with the same area.

One asks for the transformation of a circular well with diameter 4 cubits into a square well.

The side of this square is claimed to be $\sqrt{(2 \cdot 4^2)}$, which is actually the diagonal of the circumscribed square. This can hardly be a copying error.

Another one deals with two wax spheres. Many parameters are missing, which could be a copying error;

but it is also supposed that the ratio between the surface and volume is the same for a sphere and a cube, which must go back to the original compilation.

Many of the classics recur:

the broken tree, tower with moat and rope, the well with a stone thrown in, the tree gradually raised from horizontal to vertical position (Gherardi lets the top follow the circle).

The two-tower problem appears in orthodox shape, not in Jacopo's cheap version, and it is solved correctly. As always without explanation.

The area of a regular pentagon is found by the correct formula for the pentagonal *number;*

this is one of the problems that is referred to as *giomatria*.

The determination of a regular hexagon with side 5 palms follows.

It begins in good order, by prescribing a division of the hexagon into six equilateral triangles.

Unfortunately the area of each of these is then claimed to be $^1/_2 \cdot 5^2$, perhaps inspired by the formula for the area of a right triangle (but pure mess is not to be excluded).

The pavilion is dealt with realistically, arguing for the correct answer from the idea that the canvas has to be cut from cloth rectangles split into triangles.

New with respect to what we have seen so far (but regularly turning up later) is a problem about the excavation of a well.

The price for a well of specific dimensions is given, that for a well of different dimensions is asked for.

It is taken into account that digging deeper is more laborious. Supposing the labour to be proportional to the depth, for the third dimension Gherardi takes the sum of an arithmetical series until the depth.

Some problems are unusually creative in their dress (not in their mathematics).

A question about the bombardment of the fortification wall around a castle by means of a *trabocco* (a siege catapult) is opaque.

It shows, however, that such attacks had to be made by trial-and-error, and it seems to presuppose that the trajectory of the projectile is linear.

One problem deals with navigation

The corresponding map is absent at least from the edition, but it must have been similar in character to the portolan charts of the time.

Some angles must be presupposed to be right, asin the portolan, since the Pythagorean Rule is used.

This corresponds to the portolan charts.

Towards the end come sundry matters that are otherwise foreign to abbacus geometry and remained so.

The appearance of scattered words in Catalan or Provençal shows the origin of the treatise that was borrowed. They have certainly not been part of Gherardi's abbacus course and may have been added by the compiler.

The mid-15th century and onward

Around 1460, three large “abbacus encyclopedias” were produced in Florence:

- the anonymous *Libro di praticha d’aresmetricha*, Vatican, Ottobon. lat. 3307 from *ca* 1458;
- the equally anonymous *Trattato di praticha d’arismetricha*, Florence, BNC, Palatino 573 from 1460;
- Benedetto da Firenze’s *Trattato di praticha d’arismetrica*, Siena, Biblioteca degl’Intronati, L.VI.47 from 1463. I spoke of it in the previous lecture.

All three are known from the autographs – that of Benedetto also from partial copies.

The former two are somewhat free descendants of a shared model, which could have been put at the disposal of the authors by their shared teacher.

Benedetto speaks about planning to write another work on geometry. The plan may never have materialized, in any case no such work is known;

but Benedetto's intention was probably to produce a replacement or counterpart of Fibonacci's *Pratica geometrie*.

The Palatino author also speaks about such intentions. Even this is has been lost.

The intention, however, was almost certainly to produce a copy of a vernacular abridged version of Fibonacci's *Pratica geometrie*.

Such a copy is indeed contained in the manuscript Florence, BNC, Palatino 577.

The Ottoboniano manuscript also contains a geometry of this kind *after* the *Libro di praticha d'aresmetricha*

None of them thus represent the abbacus-geometry tradition.

However, the Ottoboniano manuscript includes a short geometry (5 folios in total) *within* the *Libro di praticha d'aresmetricha*.

We may take it to represent the abbacus-geometry tradition as it looked at the moment.

Then what do we find here?

Not much which is new, even though the compiler seems to have tried his own hand a couple of times, with mixed luck. On the whole he probably copies.

- A *piazza* to be paved with rectangular bricks.
- The number of bricks needed for a wall.
- The comparison of two chests, with the innovation that they are not cubic but of dimensions $4 \times 3 \times 2$ to $3 \times 2 \times 1$, which eliminates the temptation to think of a fraud.
- A stone of $4 \times 3 \times 2$ cubits costs 30 *fiorini*, how large a stone [in the same proportions] can be bought for 60 *fiorini?* First the total larger volume is found from the smaller one by means of the Rule of Three; next the sides, which involves cubic roots.

- The chord-and-arrow problem, here speaking of a partially buried millstone.
- An analogue of the pavilion problem, here dealing with a mantle.
- Transformation of the breadth of cloth of given area.
- A metal wheel with diameter 12 cubits is to be divided into three concentric circular parts.
- The cloth of two sacks of the same height, one containing 8 and the other 18 *staia*, has to be sewed together as one.

  Tacitly using that the diameters are in ratio $\sqrt{8} : \sqrt{18}$ and (still tacitly) combining with the identity (familiar in abbacus algebra since the early 14th century, in the Arabic world since al-Karajī)

$$\sqrt{8}+\sqrt{18} = \sqrt{8+18+2\sqrt{8\cdot 18}} \ ,$$

  the contents of the large sack is said to be $\sqrt{(8+28+2\sqrt{8\cdot 18})} = \sqrt{50}$ (correct if we consider the three sacks as cylinders and forget about the top and the bottom.)

– One cask, made from 30 barrel staves, contains 30 *bariglie* of wine, the contents of another one made from 20 staves (understood to be of the same length) is found as $^{400}/_{900}$ of the former cask, that is, $^{4}/_{9}\cdot 30 = 13^{1}/_{3}$ *bariglia*.

Even this is traditional. It appears in Giovanni de' Danti's *L'arte de la geumetria* from 1370.

– 4 cubits of rope binds 2 bundles of straw, how much is bound by 10 cubits?

Mathematically analogous to the previous question, but marred by wrongly written numbers suggesting sloppy copying.

– Florence is supposed to be a circle with perimeter inside the wall equal to 5 *miglia*. The thickness of the wall is $3^1/_2$ cubits, and width of the moat is 20 cubits.

The perimeter outside the moat is claimed to be $(20+2\cdot3^1/_2)\cdot3^1/_7$ cubits longer than the inner perimeter instead of $(2\cdot20+2\cdot3^1/_2)\cdot3^1/_7$ cubits.

– Florence and Prato are both supposed to be circular, with perimeters 7 and 2 *miglia*, respectively. Therefore Prato can be contained $49\div4 = 12^1/_4$ times within Florence.

- Florence is still supposed to be circular with perimeter 7 *miglia*, while Pistoia is square with perimeter 6 *miglia*.
  - Florence's area is found as if 7 *miglia* had been the diameter, becoming $38\frac{1}{2}$, while that of Pistoia is found correctly as $(1\frac{1}{2})^2$.
  - Therefore, Pistoia (obviously much larger than Prato in the previous problem) can be contained $17\frac{1}{9}$ times within Florence
  - the compiler or his original seems to have acted blindly and without thinking about the absurdity of the wrong result.

- Comparison of two well excavations. As Gherardi, the present calculator uses an arithmetical series as a way to take into account the increase of labour with increasing depth.
- Reversal of this problem type: the cost of the second well is given, and its depth is found correctly.
- The three standard problems about a tower, a moat and a rope (here, a ladder).
- A tree which is felled, falling 1 cubit per day. The top is supposed to follow the circle.

- Two problems about cisterns with a stone falling into it.
  - In the first, the cistern is full, and the volume of the stone equals that of the water flowing out
  - In the second, the cistern is not full, and it is calculated how much the water raises (the stone is wholly covered).
- Another cistern problem, but now the stone will be only partially covered before water starts flowing out. First-degree algebra with unknown *chosa* is used to find out how much is covered. (Not needed, seemingly a way to display simple algebraic ability)
- Ladder against a wall, the more difficult situation where the descent and the amount the ladder slides out are given.
  - Accompanied by a lettered diagram, and solved by means of algebra.
- Two more ladder–wall problems, where the data are packed into more complicated arithmetic, but which are geometrically similar.

  also with lettered diagrams and using algebra to get around the arithmetical obstacles.

The geometric chapter ends with the observation that infinitely many more ladder-wall questions can be formulated

> obviously only with regard to the arithmetical complications, the geometry is fully exhausted by the three familiar variants.

So, almost everything properly geometric is in the tradition as it had looked for 150 years or so. The two sacks are the exception.

The Palatino encyclopedia contains no geometrical problems, only a list of rules.

The Ottoboniano geometry is therefore likely to have been absent from the shared model (also because the general mathematical level of this model is immensely better that what we find in the Ottoboniano geometry).

We may surmise that the Ottoboniano writer, wanting to make his *Libro di praticha d'aresmetricha* complete, grasped a geometry chapter from another abbacus treatise and copied it uncritically.

Two abbacus treatises from the same epoch might make us expect something more:

Benedetto's *Tractato d'abbacho*, later than his great *Trattato di praticha*,

and Filippo Calandri's *Trattato di arithmetica*, originally written as a private school-book for Giuliano de' Medici around 1465, good enough to be printed in [1491] and again in 1518.

As a matter of fact, there is not much new to draw from either.

Benedetto includes geometry in his last chapter since without that the work "would seem imperfect".

Outside the beaten path as we know it he starts with general explanations – even more broadly discursive than those of Euclid. As an example may serve what is said about the point:

> A point is that of which no part can be taken, that is, a certain limit that cannot be drawn; but is has, as said, to be imagined by the intellect.

The π-value $3^1/_7$ is said to have been proved by Archimedes.

> Benedetto certainly knew better – he knew, translated and used difficult matters from the Latin original of Fibonacci's *Pratica geometrie*, where a true Archimedean treatment can be found.

In the problems there is nothing spectacularly new, even though for a few of the rules arguments are given – yet not for the two sacks sewed together.

In the very end, however, we find an intruder, namely an instance of *altimetria:*

> how to "measure the height without going there"

for instance of the Palazzo de' Signori, first explained in general terms and then with a numerical example.

Even Calandri finds it convenient to insert some geometry, and even he starts with some kind of definitions – more concise than those of Benedetto, and clearly independent.

About the point and the line, Calandri explains that

> the point is the limit [*termine*] of length, and the line is a length without width delimited by two points.

At its better level, abbacus geometry was thus by now expected to be founded on explanatory definitions.

We may guess that abbacus teachers in earlier times had given such explanations (probably less Euclidean) in their oral expositions.

Even Calandri's *problems* offer little new.

Noteworthy, however, is the determination of the volume of a wine tub (*tino*) shaped as a truncated cone, which is calculated as the height times the mid-cross-section.

Later, however, the volume of a conical heap of grain is calculated correctly.

The contrast leaves little doubt that the latter formula was borrowed from outside the tradition without understanding, while the former may be an improvisation based on failing intuition.

Even Calandri offers the two sacs sewed together – apparently a new fashion (the sole) of the century.

So, there was little change in the abbacus geometry tradition between Jacopo and the 1460s. How did it look in the early 16th century?

A good perspective is provided by Girolamo Tagliente's *Libro dabaco che insegnia a fare ogni raxone marcadantile, et a pertegare le terre con larte della Geometria*, "Abbacus Book That Teaches to Make All kinds of Merchantry Calculations and to Measure Terrains by the Art of Geometry",

This work was first printed in 1515 and reprinted (often pirated) at least 30 times.

It is written according to the principle “keep it simple”, for instance avoiding such difficult matters as reduction of fractions – a central topic in proper abbacus teaching.

It is profusely illustrated by woodcuts – all in all, a coffee-table book, “abbacus mathematics made easy, beautiful and entertaining”.

Tagliente’s book thus shows us what kind of geometry was supposed to belong *as a minimum* within an abbacus treatise.

As in Benedetto's and Calandri's treatises, Tagliente at first offers explanations of points, lines, surfaces and bodies, inspired by Euclid's definitions.

They are followed by an explanation that measure concerns squares; rectangles; equilateral, scalene and right triangles; circles; trapezia. Further, that metrologies vary with the location.

The problems present little new:

- The area of a square with given side.
- The area of a rectangle with given sides (three instances).
- The area of a right triangle, explained to be half a rectangle (for once an explanation!).
- Measurement of the height of a tower by means of its shadow and the shadow of a stick – similar to what Benedetto does though not identical.
- Finding the perimeter of a circle with given diameter.
- Finding the area of a circle with given diameter.
- A table of squares n^2, $2 \leq n \leq 10$, the possibility to continue being pointed out.
- A rule for finding the height of an equilateral triangle from the side, without any numerical example (a necessary omission since approximation is not explained).

– Finally three wax spheres, with perimeters 2, 3 and 6 cubits, to be merged into a single sphere.

The resulting perimeter is claimed to be $\sqrt{(2^2+3^2+6^2)}$, which would be correct if three *cylinders* with identical heights had been merged.

Spatial intuition had not improved since Jacopo.

A problem about merging two wax *cylinders* is found (and solved correctly) in a Latin geometry from the late 15th century, now in Munich and probably from German area, and otherwise not related to abbacus geometries.

This is what suggests that this rare problem type was borrowed laterally into the abbacus tradition at various moments.

This is quite meagre, as could be expected in a book intended to make mathematics easy.

It still confirms that the advent of printing (which had reached abbacus mathematics already in 1484) had not changed the expectations as to what belonged within abbacus geometry.

The main change had already occurred before the first printing of abbacus material, namely the idea to begin with general definition-like explanations.

So much about what was contained in abbacus geometry.

We may also ask, what was *absent*, beyond demonstrations, arguments and more advanced mathematics.

When writing about geometry, abbacus writers turn out to agree with what Laplace answered Napoleon asking what had become of God in his *Système du monde*:

“I did not need that hypothesis”.

This may be noteworthy. Many abbacus books contain routine references like “In the name of God” (nothing more sincere);

in the Ottoboniano *Libro di praticha*, for instance, most chapters *but not the geometry* close with a divine invocation.

Why this stasis?

We may wonder why abbacus geometry, since its stabilization as a tradition in the early 14th century, remained static, neither unfolding nor regressing

(Tagliente's regress is due to his editorial aim and was not general).

Formulated in that way, in context-free generality, the puzzle is a pseudo-problem, generated by the habit of historians of science or mathematics to concentrate on processes of change, preferably of progress.

The dynamics of intellectual progress is indeed Aristotelian, not Newtonian.

Intellectual life is part of social life, and if nothing in the general ambience provides a pushing force, then motion stops – exactly as the motion of stone on the ground.

Impetus, if such there be, is exhausted with time, as Philoponos knew, not conserved like Newtonian inertia.

Saint Jerome (the translator of the Latin Bible) could complain that in his epoch only a few idle old men “know the books, or even the name of Plato”;

after Jerome’s and Augustine’s death, nobody (before Boethius) remembered enough to complain.

But in the actual case, the puzzle is real.

The abbacus environment was not generally stagnant.

Its *algebra* started as purely rhetorical algebra with a single variable in Jacopo's *Tractatus*, but soon went far beyond that. Why algebra but not geometry?

Not because algebra was practically useful, and geometry not. Quite the contrary.

As I have said, some abbacus masters were engaged in urban surveying. Here, however, little beyond the calculation of rectangular or near-rectangular areas was needed.

Algebra, on its part, had nothing to offer to real practice, commercial or otherwise.

It served in competitions for positions or students;

here, what could impress municipal authorities or the fathers of prospective students was the ability to solve complicated problems presented by competitors,

and to present competitors with problems that went beyond *their* skill.

Initially (and for long), presenting false solutions to irreducible cubic and quartic equations could be used for that purpose;

soon, however, more perspicacious minds managed to do better by means of polynomial algebra (kept as a business secret).

As mostly happens in such situations, many of those who were engaged in this process also took private pride and pleasure, and so did some *dilettanti*.

This provided the push that moved the algebraic stone (and also caused abbacus algebra to be unable to make the final leaps toward Modern algebra).

For this, as I told in my previous lecture, the interaction with the norms of university mathematics in *Rechenmeister* algebra was a necessary though insufficient condition.

Benedetto developed symbolic first-degree algebra with up to five unknowns. It was forgotten, and had to be reinvented by Michael Stifel 80 years later

> (without impressing even then more than two or three followers before Viète and later Descartes saw how the idea could be unfolded).

Tartaglia knew how to solve certain third-degree equations and wanted to use this insight in competitions for positions;

Cardano made the knowledge public, and generalizable.

Geometry was different.

At least as a general rule, abbacus masters did not know their Euclid.

But they knew that the knowledge they applied was due to “philosophers” or could be found in *li libri di giometria*.

That hurdle was too high for anybody who might wish to create new knowledge (be it fake or genuine).

It would also call for problems which the audience of competitions might find incomprehensible or uninteresting (such as the problems dealt with by Viète and Descartes).

There was no push for that stone.

There was certainly Fibonacci.

An extract from Antonio de' Mazzinghi quoted in the *Ottoboniano* manuscript speaks about many Florentine citizens possessing Fibonacci's works (in the later 14th century).

Whether in Latin or in vernacular Antonio does not say.

In any case we know that the *Liber abbaci* was translated (badly) into vernacular already in the 13th century. We have the traces in the *Livero de l'abbecho*.

Barnabas Hughes has described three closely related vernacular versions of the *Pratica geometrie*, which must descend from a single 14th-century archetype.

A distinct family has at least two members, the Palatino-577 manuscripts and the geometry making up the last part of the Ottoboniano manuscript.

An apparently third member of this family was used by Luca Pacioli for part of the geometry of his *Summa* in 1494.

So, several vernacular translations of Fibonacci's *Pratica geometrie* circulated.

But they omitted the most advanced parts, and no complete versions seem to have been at hand at least in Florence:

> in order that his treatise "may have no imperfections", Benedetto copies and translates himself Fibonacci's finding of two mean proportionals in his *Trattato di praticha*.

Even the availability of the reduced *Pratica geometrie* in vernacular and the full text in Latin thus did not provide inspiration or a starting point for further development

- and there was no connection forward to the creative Italian geometers of the 16th century.

Francesco Maurolico and Federico Commandino had no use, neither for Fibonacci nor, a fortiori, for abbacus geometry.

Not because they were prudish as a Jean Borrel and a François Viète

- Maurolico wrote *Demonstrationes algebrae* without any qualms;
- and Commandino was said in an early biography to have planned to publish updated versions of Fibonacci's as well as Pacioli's work, apparently as acts of Humanist piety (though without doing so).

But they knew what was worthwhile and what was past expiry date.

Their inspiration was not even Euclidean (although they knew and used their Euclid) but Archimedes.

Johannes Regiomontanus had been eager in the 1460s to learn Florentine algebra; but his geometry was Euclidean, in part directly, in part indirectly.

So, in conclusion:

Abbacus geometry, even together with its subordinate Fibonaccean follower, turned out to be

not only a tradition without pretensions
but also a tradition without a future

quite different in the latter respect from the basic abbacus syllabus, which I, like millions of others, still encountered in school in the 1950s.