Jens Høyrup

Traces of a forgotten mathematical culture in 12th-century al-Andalus

Lecture at

Tsinghua University

Beijing, 15 November 2024

In my first three lectures, I have mainly dealt with some of those mathematical cultures which belong to the canon of the historiography of mathematics:

– Greek and Hellenistic Antiquity, the backbone of the historiographic tradition since its very beginning

– and Mesopotamia, whose Babylonian segment became part of it a century ago.

Today I shall deal with the indirect evidence we have for a mathematical culture whose very existence has gone unnoticed, and which will hardly ever be adopted into the canon since direct evidence has vanished.

This culture is that of 12th-century al-Andalus.

At the ʿAbbāsid revolution in 750 CE, an independent caliphate had been established in Córdoba in al-Andalus (Islamic Spain) around an accidental survivor from the Umayyad dynasty.

The Umayyad Córdoba caliphate collapsed at the beginning of the 11th century, and Al-Andalus dissolved into a number of small kingdoms.

the northern part of the Iberian Peninsula was already under Christian rule, sometimes submitting to the caliphate, sometimes fighting it.

Toward the end of the 11th century, the Muslim kinglets were forced by Christian military pressure to call in the Berber Almoravids, who already ruled most of present-day Algeria and Morocco.

After a few decades this dynasty lost its power, in the Maghreb to the Almohads (a religious reform movement, equally of Berber origin), in al-Andalus to local kinglets, who however were called to order around 1150 by the Almohads.

As the Almohad dynasty broke down under Christian pressure and in internal strife, a gradually dwindling al-Andalus once again organized in smaller states – Valencia, Murcia and Granada, the last of which fell famously in 1492 to the "Catholic Kings" Isabella and Ferdinand.

The most outstanding representative of 12th-century intellectual life in al-Andalus was the philosopher ibn Rušd – in Latin Averroës, but often simply *Commentator* because of his immensely influential commentaries to Aristotle.

Influential, however, only in Latin and Hebrew philosophy, but leaving modest traces in the Islamic world outside al-Andalus, which moreover mainly knew him through the Jewish philosopher Maimonides.

Not all currents of al-Andalus thought from the same century were isolated – Ibn al-Yāsamin, plausible inventor of the Maghreb algebraic notation" (on which briefly in the next lecture) moved between al-Andalus and Morocco
– but then Maghreb algebra was relatively isolated.

The astronomical works of Jābir ibn Aflaḥ from the first half of the 12th century were also very influential in the Latin world, but seem to have influenced the rest of the Islamic world slightly – and once again mainly through Maimonides.

In contrast, al-Muʾtaman ibn Hūd's 11th-century *Kitāb al-Istikmāl* ("Book of Perfection") had given rise to further work by Arabic mathematicians.

His work, and what we know indirectly about the works of the contemporary Ibn Sayyid, made Ahmed Djebbar conclude in 1993 that there was

> in Spain and before the 11th century, a solid research tradition in arithmetic whose starting point seems to have been the translation made by Thābit ibn Qurra of Nicomachos' *Introduction to Arithmetic*.

Over the last two decades I have run into a number of sources that indicate beyond reasonable doubt that this tradition, or at least something related, was still alive and fertile in al-Andalus in the 12th century

during which some of its products were in part translated, in part paraphrased in Latin works.

The arguments that they point to al-Andalus are sometimes rather technical, I must warn. I shall abbreviate as much as I can in the oral presentation but leave the technicalities in the presentation for reading.

The unknown heritage

My first topic is a theoretical elaboration of the solution to a stunning recreational problem – the "unknown heritage".

But first as background something about this recreational problem itself:

The standard version of that problem runs as follows: a father leaves to his first son 1 monetary unit and $\frac{1}{n}$ (n usually being 7 or 10) of what remains; to the second he next leaves 2 units and $\frac{1}{n}$ of what remains, etc.

In the end all sons get the same amount, and nothing remains. The solution is that there are n–1 sons, each of whom receives n–1 monetary units.

> Alternatively the fraction is given first and the arithmetically increasing amount afterwards, in which case n–1 sons get n monetary units each.

Our earliest source for the problem is Chapter 12 of Fibonacci's *Liber abbaci*.

It is possible to find the only *possible* solution by algebra or by a double false position from the equality of the first two shares.

In order to show that this really *is* a solution one next has to calculate stepwise.

This "simple" problem is found regularly in Italian abbacus books from the early 14th century onward, mostly the first variant but

– also sometimes the second,
– and the corresponding semi-simple variants where the absolute contributions are not i (i = 1, 2, ...) but $i\varepsilon$ (i = n, n+1, ...), which corresponds to taking ε and not 1 as the monetary unit and skipping the first n heirs.

Such semi-simple variants are also presented by Fibonacci.

Most abbacus authors merely state the solutions, occasionally we find a numerical check or use of the double false position.

Of particular interest is,

- firstly, the appearance of the problem in Maximos Planudes's Byzantine late 13th-century *Calculus according to the Indians, Called the Great;*

- secondly, the *absence* from known Arabic sources (even though two contain a derived and simplified version).

First, Planudes presentation:

– the problem follows just after the explanation of how to calculate with Hindu-Arabic numerals,

– and comes just before discussion of a problem in almost Diophantine style, for a given n to find two rectangles ⊏⊐(a,b) and ⊏⊐(c,d) such that $a+b = c+d$, $n \cdot ab = cd$ (a, b, c and d being tacitly assumed to be integers).

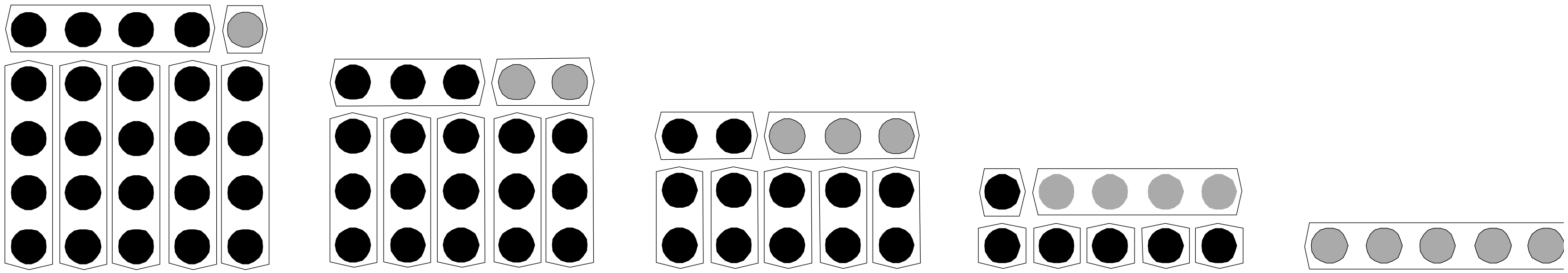
This problem also appears in pseudo-Heronian *Geometrica*-manuscripts, and is thus of late ancient or early Byzantine date.

The inheritance problems serves as illustration of this theorem:

> When a unit is taken away from any square number, the left-over is measured by two numbers multiplied by each other, one smaller than the side of the square by a unit, the other larger than the same side by a unit.
>
> As for instance, if from 36 a unit is taken away, 35 is left. This is measured by 5 and 7, since the quintuple of 7 is 35.
>
> If again from 35 I take away the part of the larger number, that is the seventh, which is then 5 units, and yet 2 units, the left-over, which is then 28, is measured again by two numbers, one smaller than the said side by two units, the other larger by a unit, since the quadruple of 7 is 28.

If again from the 28 I take away 3 units and its seventh, which is then 4, the left-over, which is then 21, is measured by the number which is three units less than the side and by the one which is larger by a unit, since the triple of 7 is 21. And always in this way.

Pebbles are not mentioned in the text;



however, a pebble argument as shown here not only allows us to understand the solution, it is also a likely basis for the discovery that the counter-intuitive problem was *possible*.

Together with other evidence it is a strong suggestion that the problem was invented in Byzantium or late Greek Antiquity.

Fibonacci also gives a veiled hint that he knows the problem from Byzantium.

Second, there are the simplified Arabic versions.

Ibn al-Yāsamīn's *Talqīḥ al-afkār fī'l ʿamali bi rušūm al-ghubār* ("Fecundation of thoughts through use of *ghubār* numerals", Marrakesh, c. 1190) brings this problem:

An inheritance of an unknown amount. A man has died and has left at his death <u>to his six children</u> an unknown amount.

He has left to one of the children one dinar and the seventh of what remains, to the second child two dinars and the seventh of what remains, to the third three dinars and the seventh of what remains, to the fourth child 4 dinars and the seventh of what remains, to the fifth child 5 dinars and the seventh of what remains, and to the sixth child what remains. He has required the shares be identical. What is the sum?

The solution is to multiply the number of children by itself, you find 36, it is the unknown sum. This is a rule that recurs in all problems of the same type.

This problem can be solved stepwise, by backward solution. None the less, the rule which is given is that for Fibonacci's problem.

The reference to “all problems of the same type” shows that ibn al-Yāsamīn must know it as a common type, in the Maghreb or in al-Andalus.

Similar backward calculations could be made for fractions that change and for absolutely defined contributions that are not in arithmetical progression. However, the rule is only valid for Fibonacci’s type.

The version offered by Ibn al-Hāʾim some 200 years later is not identical, but still with given number of heirs, and still solved by the rule we know from Fibonacci.

We may conclude that the Arabic problem descends from the “Christian” problem and results from an unfelicitous attempt to assimilate it to the more familiar “Chinese box” structure. Fibonacci and Planudes show us the original.

The sophisticated version

Fibonacci, however, present us with more, and that is where things become interesting for our topic.

A notation will be handy.

Division of a number where each of the successive parts receives first $\alpha+i\varepsilon$, $i = 0, 1, 2, \ldots$, and afterwards ϕ times what remains at hand ($\phi = {}^{p}/_{q} < 1$, but not necessarily an aliquot part $\frac{1}{n}$) we shall designate $(\alpha,\varepsilon|\phi)$;

a division where instead ϕ times what is available is taken first, and afterwards an absolutely defined amount $\alpha+i\varepsilon$, we shall designated $(\phi|\alpha,\varepsilon)$.

In this notation, Fibonacci offers the following problems:

$(1,1\|\frac{1}{7})$	$(1,1\|\frac{2}{11})$	$(2,3\|\frac{6}{31})$	$(3,2\|\frac{5}{19})$
$(\frac{1}{7}\|1,1)$	$(4,4\|\frac{2}{11})$	$(\frac{6}{31}\|2,3)$	$(\frac{5}{19}\|3,2)$
$(3,3\|\frac{1}{7})$	$(\frac{2}{11}\|1,1)$		
$(\frac{1}{7}\|3,3)$	$(\frac{2}{11}\|4,4)$		

The column to the left contain the two variants of the simple version, with the equally simple variant that the monetary unit is 3 *bizantii* instead of 1.

Here, everything is stated in terms of a father distributing his possessions to his sons.

The remaining columns speak about dividing a number or a number of *dragmae* (serving much as Diophantos's *monades*) in the ways indicated (thus eschewing fractional "sons").

Here, all shares are similarly stated to be equal, with the difference, however, that the last share may be fractional.

The problems in the second column are dealt with according to the rule for the simple version, with the unexplained trick that $\frac{2}{11}$ is understood as $\frac{1}{5\frac{1}{2}}$. Then the number of shares turns out to be 4½, meaning that the last (half-)share is only half of the others.

This trick does not work in the third and the fourth column. For the problem $(2,3|\frac{6}{31})$ Fibonacci finds the only possible solution by means rhetorical first-degree algebra: he calls the number to be divided a *thing*, computes the first two shares and equates them.

The number to be divided is then found (if expressed in a formula that follows the calculations step by step) to be

$$T = \frac{q^2(\alpha+\varepsilon)-(q-p)q\alpha-(q-p)p\alpha-(\alpha+\varepsilon)pq}{p^2} \quad . \tag{1}$$

Fibonacci afterwards makes a step-by-step calculation, showing that all shares are indeed equal.

He may be aware that his algebraic calculation merely finds the *only possible* solution.

In the end he claims to extract from the calculation this rule:

$$T = \frac{[(\varepsilon - \alpha)\, q + (q - p)\, \alpha] \cdot (q - p)}{p^2} \; , \tag{2a}$$

$$N = \frac{(\varepsilon - \alpha)\, q + (q - p)\, \alpha}{\varepsilon p} \; , \tag{2b}$$

$$\Delta = \frac{\varepsilon\, (q - p)}{p} \; , \tag{2c}$$

N being the number of shares and Δ the value of each of them.

He cheats. With techniques at Fibonacci's disposal (1) cannot be transformed into (2a).

We may conclude that Fibonacci took over a rule whose derivation he did not know, and claimed it to be a consequence of his own solution.

This is confirmed by the solutions to the other sophisticated problems.

I know of no surviving text from the 14th or the early decades of the 15th century where these sophisticated versions of turn up.

They are copied from the *Liber abbaci* in three “abbacus encyclopedias” written in Florence around 1460

The next trace in Italy – but a mere trace – is then in Cardano’s *Practica arithmetice et mensurandi singularis* from 1539. It is corrupt but shows that there was some broader circulation.

This is confirmed by Barthélemy de Romans’ *Compendy de la praticque des nombres*, a Franco-Provençal treatise from c. 1467,

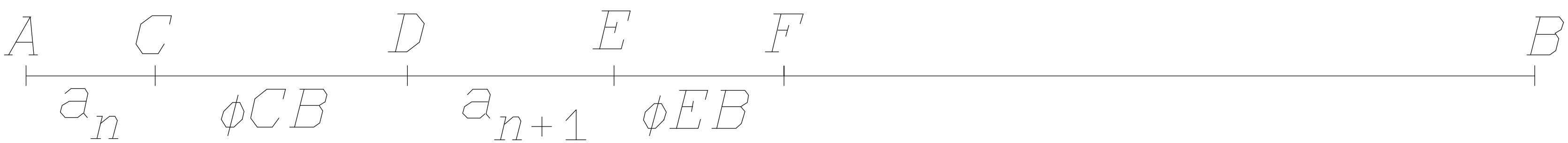
It is independent of Fibonacci but knows all the sophisticated variants.

Summing up all evidence;

The sophisticated variants must have been invented in Provence, or the Iberian world.

Given the lack of any evidence that 12th-century Provence or Christian Spain should have been home to mathematicians with the required level of competence, al-Andalus seems to be the cradle.

Fibonacci's algebra does not show us how the sophisticated variants were found provable. But elsewhere he shows us a technique by means of which this could be done: the line diagram, well known in al-Andalus.



Liber mahameleth

My next item is *Liber mahameleth*, an anonymous Latin work that was discovered by Jacques Sesiano in 1974 and first described by him in 1988;

more recently, two critical editions have appeared, prepared respectively by Anne-Marie Vlasschaert in 2010 and by Sesiano in 2014, the latter with translation.

Both editors agree that the title refers to Arabic *al-muʿāmalāt* – that is, socially (mostly commercially) useful mathematics.

Sesiano proposes the compilation to have been made by John of Seville, taking a 15th-century reference to the author as *ispano*, “Spanish”, as a mistake for *hispalensis*, “Sevillian”,

but also because the *Liber mahameleth* shares a number of passages with John’s *Liber algorismi*. He also points to evidence that the work was written in a Muslim environment (thus, as he sees it: before John left Seville).

However, the strong discrepance between the treatment of algebra in the two works excludes the authorship of John.

Charles Burnett, because of more shared passages with Gundisalvi’s *De divisione philosophiae*, points to Gundisalvi or his contact John of Spain (who may be the same as Joohn “of Seville”, and in any case author of the *Liber algorismi*). as the author.

But, firstly: nothing suggest either of the two to have possessed the mathematical level of the *Liber mahameleth.*

Secondly, Gundisalvi, in *De divisione*, describes the contents of a book "in Arabic called *liber mahamalech*", which coincides with the contents listed in the *Liber mahameleth*.

> To be observed: in the writing of the time, «c» is almost indistinguishable from «t».

So, Gundisalvi or a collaborator of his almost certainly made a (probably free) translation of an existing book from al-Andalus, not a compilation from many sources.

Sesiano believes that a reference to “what the Arabs do” shows that the writer cannot himself have been an Arab.

However, it is a close parallel to a passage in Gundisalvi’s translation of al-Fārābī’s *Catalogue of the Sciences*, so if anything it is supplementary evidence that the treatise was translated by Gundisalvi himself or somebody from his group.

Of interest for our topic is the way commercial problems serve as pretext for advanced experimentation.

If p and P stand for prices and q and Q for the appurtenant quantities, we have $\frac{q}{p} :: \frac{Q}{P}$ (this is meant as a proportion, not an equation involving two fractions).

As we know, this implies the "product rule" $q \cdot P = p \cdot Q$.

Some of the sophisticated variants are:

$\frac{3}{13} :: \frac{Q}{P}$, $Q+P = 60$. This is solved by means of proportion theory, namely via transformation into $\frac{3}{3+13} :: \frac{Q}{Q+P}$ and subsequent use of the product rule.

$\frac{3}{13} :: \frac{Q}{P}$, $P–Q = 60$. Similarly. Such systematic variation is pervasive.

$\frac{3}{8} :: \frac{Q}{P}$, $Q{\cdot}P = 216$. Fractions (or the rules that $\frac{P}{Q} :: \frac{PQ}{QQ} :: \frac{PP}{PQ}$) are not mentioned, but the solution that is offered builds on awareness that

$$(3{\cdot}216)\div 8 = \frac{3}{8}{\cdot}216 = \frac{Q}{P}{\cdot}(Q{\cdot}P) = Q^2$$

and

$$(8{\cdot}216)\div 3 = \frac{8}{3}{\cdot}216 = \frac{P}{Q}{\cdot}(Q{\cdot}P) = P^2$$

$\frac{4}{9} :: \frac{Q}{P}, \sqrt{Q}+\sqrt{P} = 7\frac{1}{2}.$

It is used but not make explicit that $\frac{\sqrt{4}}{\sqrt{9}} :: \frac{\sqrt{Q}}{\sqrt{P}}$, which is no standard theorem from the theory of proportions but follows easily from an arithmetical understanding.

From here as previously.

Alternatively,

$$\sqrt{\frac{4}{9}}+1 = \frac{\sqrt{Q}}{\sqrt{P}}+1 = \frac{\sqrt{Q}+\sqrt{P}}{\sqrt{P}} = \frac{7\frac{1}{2}}{\sqrt{P}} ,$$

which also presupposes an underlying arithmetical conceptualization.

Yet another alternative makes the claim that

$$\left(\sqrt{\frac{(\sqrt{P}+\sqrt{Q})^2}{(P-Q)/Q} + \left(\frac{\sqrt{P}+\sqrt{Q}}{(P-Q)/Q}\right)^2} - \frac{\sqrt{P}+\sqrt{Q}}{(P-Q)/Q} \right)^2 = Q ,$$

– which is true but not easy to see or even verify, in particular not if not expressed in modern symbolism. The text does not explain.

A chapter follows “about the same, with [algebraic] things”.

When *res* and *census* appear in the text I shall render them by r and C, respectively.

First $\frac{3}{10+r} :: \frac{1}{r}$

This is transformed into $\frac{3}{10+r} :: \frac{3}{3r}$, whence $3r = 10+r$, which is solved in the usual *al-jabr* way.

Alternatively, the proportion is transformed into $\frac{3-1}{(10+r)-r} :: \frac{1}{r}$, that is, $\frac{2}{10} :: \frac{1}{r}$, whence $\frac{1}{5} :: \frac{1}{r}$, etc.

As we see, cross-multiplication is not used to establish the equation; instead the antecedents are made equal, whence the consequents also become equal. This preference is general.

In “another chapter about an unknown in buying and selling” then follows:

> An unknown number of measures is sold for 93, and addition of this number to the price of one measure gives 34

– in our symbols (since no *res* occurs): $x + \frac{93}{x} = 34$.

At first the solution is given as

$$\frac{34}{2} \pm \sqrt{\left(\frac{34}{2}\right)^2 - 93} \ ,$$

the sign depending on whether the number of measures exceeds or falls short of the price of one measure.

Next a geometric argument based on the principles of *Elements* II.5 is given.

Euclid is not mentioned, however, which the compiler-author is elsewhere fond of doing;

the direct or indirect inspiration might be Abū Kāmil's similar proof for the fifth *al-jabr* case – elsewhere it is clear that the author knew Abū Kāmil well.

In any case, it reminds of the "key" versions of the *Elements*-II propositions which we shall encounter later on, and is thus evidence of some kind of relationship.

Next comes the first of the two corresponding subtractive variants, namely the one in which the number of measures subtracted from the price of one of them gives 28.

First a numerical prescription is given, next a line-based geometric proof.

If instead (the second subtractive variant) subtraction of the number of measures from the price of one of them gives 28, one should proceed correspondingly.

Then follows

$$\frac{q}{p} :: \frac{Q}{P},\ pq = 6,\ PQ = 24,\ (p+q)+(P+Q) = 15.$$

Once again the argument appears to go via the factor of proportionality s, $sp = P$, $sq = Q$ – as this time a geometric argument confirms.

$\frac{q}{p} :: \frac{Q}{P}$, $pq = 10$, $PQ = 30$, $(p+q)+(P+Q) = 20$.

This seemingly innocuous variation of the preceding question leads to an irrational value $s = \sqrt{3}$, and therefore to complications and a cross-reference to the chapter about roots (where indeed the necessary explanations are found).

In the end, this leads to a discussion in terms of the classification of *Elements* X (not mentioned here, which suggests that these classes are supposed to be familiar – elsewhere the book *is* mentioned).

Etc.

This is clearly written by somebody who is at least as familiar with proportion techniques as with *al-jabr*, and furthermore familiar with *al-jabr* through Abū Kāmil.

What we are presented with is *muʿāmalāt* “from a higher vantage point” (to quote Felix Klein), not simply a presentation of *muʿāmalāt* mathematics for a Latin readership.

And it is far beyond the mathematical level of Gundisalvi and his circle, indeed above anybody who had not *practised* proportion manipulations as done for instance in mathematical astronomy.

This time, there is no need to make much detective work on the basis of indirect evidence. What we have is a translation of something written in al-Andalus and then translated into Latin in Toledo around 1160

Back to Fibonacci

Liber abbaci Chapter 15 falls into three sections, all relevant to our purpose. At first we shall look at section 1.

Chapter 15 as a whole is said to treat of "geometrical rules, and questions of algebra and almuchabala".

Section 1, on its part, is said to deal with “proportions of three and four quantities, to which many questions pertaining to geometry are reduced”.

Actually, the text speaks consistently of *numbers*, not quantities; moreover, the results are rarely (and never explicitly) used in the ensuing “geometry” section where they might have served.

Fibonacci’s initial announcement is out of keeping with what follows.

The chapter heading and the subheading contradict each other in a way that shows part 1 to have been added in the 1228 version, without corresponding editing of the heading.

Everything in part 1 deals with proportions involving 3 or 4 numbers, as the subheading says.

When three numbers are involved, Fibonacci refers to them as minor/middle/major, first/middle/major or first/second/third; if four, as first/second/third/fourth. For convenience, I shall use $P/Q/R$ respectively $P/Q/R/S$.

Mostly, the problems are accompanied by letter-carrying lines drawn in the margin.

First come proportions involving three number, afterwards a few questions involving four numbers.

Using conjunction, disjunction, permutation etc. (all coming from *Elements* V), Fibonacci transforms the given proportion in such a way that the numbers can be found from the product rules by means of addition or subtraction or, more often, *Elements* II.5–6 (in numerical or "key" version – I shall return to what that means).

Fibonacci never refers to Euclid here, as is his habit elsewhere, but only uses line diagrams. That is, he uses the “key” versions.

Since the omission is systematic, we may already at this point be confident that his use was indirect and the material thus borrowed.

The section can be divided into 50 propositions, most of which are problems:

#1–3 deal with three numbers in continued proportion, $P : Q : R$ (that is, $\frac{P}{Q} :: \frac{Q}{R}$) of which one and the sum of the other two are given. The naming of segments presupposes the Latin alphabetic order *a*, *b*, *c*, ….

---*Incipit pars prima*---

#1 $P+Q = 10$, $R = 9$. *Conjunctim* $\frac{P+Q}{Q} :: \frac{Q+R}{R}$, whence *Elements* II.6 can be applied to $Q\cdot(Q+9) = 90$, which follows from the product rule.

#2 $P = 4$, $Q+R = 15$. Analogous.

#3 $Q = 6$, $P+R = 13$. The product rule gives $P\cdot R = 36$, which is transformed so as to permit use of *Elements* II.6 (direct use of II.5 seems obvious).

#4–38 still treat of three numbers, but now differences between the numbers are among the given magnitudes.

The alphabetic order changes to *a*, *b*, *g*, *d*, …, pointing to use of a Greek or an Arabic source (perhaps in pre-existent Latin translation).

However, in #4–5, still dealing with a continued proportion, *c* is made use of in the calculations:

#4 $P : Q : R$, $Q{-}P = 2$, $R = 9$. *Disjunctim* $\frac{R}{Q} :: \frac{R-Q}{Q-P}$. Solved by means of *Elements* II.5.

#5 $P : Q : R$, $R{-}P = 5$, $Q = 6$. The product rule gives $P{\cdot}R = 36$, which allows use of *Elements* II.6.5.

#6 An aside which explains that $\frac{a}{b} :: \frac{c}{d}$ entails that the squares of the numbers are also in proportion.

---Modus alius proportionis inter tres numeros---

#7 $\frac{R-Q}{Q-P} :: \frac{R}{P}$, Q unknown. $R-P$ thus has to be split into two parts having the ratio $R : P$; this is solved as a partnership problem (the link is not made explicit).

#8 Same proportion, R unknown. *Permutatim* $\frac{R}{R-Q} :: \frac{P}{Q-P}$, a first-degree problem.

#9 Same proportion, P unknown, solved similarly.

---Modus alius proportionis inter tres numeros---

#10 $\frac{Q-P}{R-Q} :: \frac{R}{P}$, Q unknown. *Conjunctim* $\frac{(Q-P)+(R-Q)}{R-Q} :: \frac{R+P}{P}$, a first-degree problem.

#11 Same proportion, R unknown. Product rule, and *Elements* II.6.

#12 Same proportion, P unknown. Analogously.

---Modus alius proportionis in tribus numeris---

#13 $\frac{R}{P} :: \frac{(R-Q)+(Q-P)}{R-Q}$, Q unknown. Since $(R-Q)+(Q-P) = R-P$, this is as simple first-degree problem.

#14 Same proportion, R unknown. $\frac{R-P}{P} :: \frac{Q-P}{R-Q}$. From the product rule follows that the product of $R-P$ and $R-Q$ is known. So is their difference, which allows application of *Elements* II.6.

#15 Same proportion, P unknown. Product rule and *Elements* II.5.

Etc.

---Incipit de proportione quattuor numerorum---

#39 From $\frac{P}{Q} :: \frac{R}{S}$ follows $\frac{Q}{P} :: \frac{S}{R}$ and $\frac{R}{P} :: \frac{S}{Q}$. From the product rule $PS = QR$, any one of the numbers can be found from the others.

#40 $P+Q$, R and S known. $\frac{P+Q}{Q} :: \frac{R+S}{S}$, whence Q.

#41 $R+S$, P and Q are known. Similarly

#42 $P+R$, Q and S known. $\frac{Q+S}{S} :: \frac{P+R}{R}$, whence R.

#43 $Q+S$, P and R are known. Similarly

#44 $Q+R$, P and S known. The product rule allows application of *Elements* II.5.

#45 Similarly if $P+S$, Q and R are known. Illustrated by an example involving *rotuli* (a weight unit) and *bizantii* (a monetary unit) and their sum.

(We are reminded of the contorted problems from the *Liber mahameleth*)

Etc.

The Latin alphabetic order of #1–3 suggests that this group comes from Fibonacci's own pen, at least in its final redaction. The purely Greek or Arabic order of #7–50 suggest a less edited borrowing.

It may well come from an earlier Latin translation. The 12th-century translators whose work we know were as faithful to the lettering of their originals as possible.

The mixed order of #4–5 (there is no lettered diagram for #6) is most likely to reflect that even these were borrowed, but the calculations reconstructed or made anew by Fibonacci.

The alphabetic order tells us that the source for #4–50 was not Latin; the various points of contact of #6–50 with the *Liber mahameleth* suggest the Iberian area and hence al-Andalus.

Pappos's *Collection* III and Nicomachos' *Introduction to Arithmetic* present us with sequences of "means" between two numbers P and R (not fully identical).

As it turns out, Fibonacci's #7–38 are very closely related to this ancient classification and theory of means. For two numbers P and R, and the mean Q, it is shown how knowledge of any two of them allows us to find the third. This table illustrates it:

	Pappos	Nicomachos	*Liber abbaci*
$\frac{R-Q}{Q-P} :: \frac{R}{R}$ (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P} :: \frac{R}{Q}$ or $\frac{R-Q}{Q-P} :: \frac{Q}{P}$	P2	N2	#27–29
$\frac{R-Q}{Q-P} :: \frac{R}{P}$	P3	N3	#7–9
$\frac{R-Q}{Q-P} :: \frac{P}{R}$	P4	N4 (but inverted)	#10–12 (inverted)
$\frac{R-Q}{Q-P} :: \frac{P}{Q}$	P5	N5 (but inverted)	#34–36 (inverted)
$\frac{R-Q}{Q-P} :: \frac{Q}{R}$	P6	N6 (but inverted)	#20–22 (inverted)
$\frac{R-P}{Q-P} :: \frac{R}{P}$	absent	N7	#16–18
$\frac{R-P}{R-Q} :: \frac{R}{P}$	P9	N8	#13–15
$\frac{R-P}{Q-P} :: \frac{Q}{P}$	P10	N9	#30–32

$\frac{R-P}{R-Q} :: \frac{Q}{P}$	P7	N10	#37–38
$\frac{R-P}{R-Q} :: \frac{R}{Q}$	P8	absent	#23–25

As we see, indeed, Fibonacci agrees with Nicomachos and not with Pappos in the cases 4–6, having $\frac{R}{P} :: \frac{Q–P}{R–Q}$ instead of $\frac{R–Q}{Q–P} :: \frac{P}{R}$, etc.

We may thus assume that the ultimate inspiration is Nicomachos, not Pappus. This is not astonishing, Nicomachos was well known in the Arabic as well as in the Latin world (in the latter in Boethius's translation).

Djebbar, as already quoted, also pointed to a tradition starting from Thābit ibn Qurra's translation of Nicomachos.

There are obvious differences, however. Firstly, the *Liber abbaci* contains no counterpart of the arithmetical mean, whose expression as a proportion is indeed next to ridiculous – the only sensible definition is $R–Q = Q–P$.

Whoever made Nicomachos' material the object of systematic theoretical exploration has seen that.

On the other hand, the case omitted by Nicomachos but discussed by Pappos is included.

This is not evidence of contamination, a competent mathematician going through all the cases would observe that it has its place

– just as he observes that Fibonacci's #26, $\frac{R}{Q} = \frac{R-P}{Q-P}$, ought to be there, at least once we have given up the idea that Q is a genuine mean and therefore $P<Q<R$.

This proportion indeed entails $R = Q$ (since $P = 0$ is out of the question)

Fibonacci actually does not speak of means, for which reason we may assume that his source did not either.

Finally, the treatment of means is extended by a similar investigation of four numbers in proportion, which because of the alphabetic ordering is likely also to represent a borrowing, presumably from the same source.

All in all: the vicinity of the methods of Fibonacci's #7–50 to those of the *Liber mahameleth*, together with the uniqueness of the two texts, makes it reasonably certain that even Fibonacci's source came,

– if not from the same hand then at least from the same environment as the theoretical exploration of the possibilities offered by *muʿāmalāt* mathematics.

On the other hand, the absence of anything in known Ibero-Latin texts similar to Fibonacci's Chapter 15 Section 1 makes it implausible that any of the two originated in the ambience of translators into Latin and the users of these

– even more implausible than the idea in itself that the original *Liber mahamalech* should have been produced by Gundisalvi or an associate of his.

10 minutes’ break

A lost 12th-century Latin translation of Arabic algebra

Chapter 14 of Fibonacci's *Liber abbaci* deals with roots, and combines what we can find in the initial part of al-Khwārizmī's algebra with material from Euclid's *Elements* X.

> It is rather messy, almost certainly because Fibonacci inserted much new material from *Elements* X when preparing the new version in 1228 without editing what was already there – but that does not concern us here.

At first, however, comes a preamble:

> Let it be me permitted to insert in this chapter about roots certain necessary matters that are called keys [*claves*]; since they are all proved by clear demonstrations in Euclid's Second, it will suffice beyond their definitions to proceed by means of numbers.

The first of which is that, when a number is divided into any number of parts, then the multiplications of these parts in the whole divided number, joined together, are equal to the square of the divided number, that is, the multiplication of the same number in itself.

[*Elements* II.1, applied to two equal lines; or, if we prefer, *Elements* II.2 generalized to division into several parts.]

For example: let 10 be divided into 2, and 3, and 5. I say that the multiplications of the two, the three, and the five in 10, evidently 20, and 30, and 50, equal the multiplication of 10 in itself, that is, 100.

[Similar versions follow of *Elements* II.1; II.4; and the corollary $2a\cdot(a+b)+b^2 = a^2+(a+b)^2$.]

> Further, if a number is divided into two equal parts, and also into unequal parts, then the multiplication of the smaller part by the larger, together with the square of the number which there is from the smaller part until the half of the whole divided number will be equal to the square of the said half
>
> > [*Elements* II.5; follows a numerical example and a similar version of II.6.]
>
> To the latter two definitions are reduced all questions from *aliebra* et *almuchabala*, that is, in the book of *contemptio* and *solidatio*.
>
> > [*contemptio* is certainly a mistake for *contentio*, "comparison/contrast/ struggle"

As shown by the present tense "are called" (*dicuntur*), the notion of "keys" is borrowed, not introduced by Fibonacci himself, who in that case would use the future tense.

The reference to *aliebra almuchabala* that closes the presentation of the keys leaves no doubt that it is adopted from an Arabic source.

However, this use of “keys” seems not to be known from extant Arabic writings – in these, the “key” is that which unlocks a subject.

We thus have to think of a region whose theoretical knowledge did not spread to the rest of the Arabic world, which once again leads us to al-Andalus;

the regular use of the “key”-version of *Elements* II.5–6 in the *Liber mahameleth* and in chapter 15 parts 1 (and occasionally 2, as we shall see) of the *Liber abbaci* confirms this inference, even though “keys” are never spoken of in any of these.

On the other hand, the translation offered for *aliebra almuchabala* is puzzling. After the emendation it is quite adequate (apart from an inversion of order)

– but wholly different from what we encounter elsewhere at the time, and also from the translation given by Fibonacci when he presents the topic.

The only credible explanation is that this translation is part of the borrowed text, and thus that Fibonacci (at least here, but quite likely also elsewhere when he borrows from al-Andalus) takes advantage of an existing Latin translation which has now been lost.

In the rare cases where we can identify his source, Fibonacci does indeed copy verbatim, while avoiding pastiche in his commentaries.

It may be worth noticing that Fibonacci includes no Iberian locations (neither al-Andalus nor Castile) when listing in his introduction the places of trade where he had learned after his boyhood stay in Bejaïa

- namely “Egypt, Syria, Greece, Sicily, and Provence”.

This might imply that what he had taken from al-Andalus came from written material, not from direct confrontation.

> In the first version from 1202 he does indeed state to copy from a Castilian treatise on barter. The passage has been removed in the 1228 version but has escaped in a single contaminated manuscript.

The copied section ends with the puzzling Latin translation. Indeed, “finished this”, the description of the contents of the chapter follows.

A few second-degree problems are solved in chapter 15 part 2 (after the transformed investigation of means and before the presentation of algebra) by means of proportion theory and "key" versions of *Elements* II.5–6.

They clearly belong to the same large family as the *Liber mahameleth* and the preamble to chapter 14.

They have nothing to do with geometry and are likely to have been inserted in the 1228 edition. Fibonacci apparently found no better place for them.

I shall not say more about them.

Borrowings from a Latin redaction of Abū Kāmil's algebra

Chapter 15 part 3 of the *Liber abbaci* contains an algebra – first the rules, then 99 questions with interspersed theory. In the second section we find two sequences of indirect borrowings from Abū Kāmil.

Explaining these, however, presupposes knowledge familiarity with some fundamentals of Arabic algebra.

The first extant Arabic introduction to algebra (and probably the absolutely first to be written down in a treatise) was written by al-Khwārizmī in the earlier ninth century CE.

Al-Khwārizmī's name for the technique is rendered *algebra et almuchabala* by Gerard of Cremona, followed by Fibonacci. This time he translates it "opposition and restoration" (with the same puzzling inversion of the order).

“Restoration” (algebra) is the additive repair of a deficiency, “opposition” originally the confrontation of two equals by which a simplified equation is established.

> With time, “opposition” was mostly used instead about the subtractive counterpart of “restoration”.

The core of the technique was constituted by six rules for basic first- and second-degree equations.

> C here stands for what in Latin became *census*, originally an amount of money (in Arabic *māl*), r for its square root (Latin *radix*), and N for number. α stands for an unspecified coefficient (implicit in the use of a plural):

Kh1	$C = \alpha r$ – first example $C = 5r$.
Kh2	$C = N$ – first example $C = 9$.
Kh3	$\alpha r = N$ – first example $r = 3$.
Kh4	$C+\alpha r = N$ – first example $C+10r = 39$.
Kh5	$C+N = \alpha r$ – first example $C+21 = 10r$.
Kh6	$\alpha r+N = C$ – first example $3r+4 = C$.

Originally these were thus supra-utilitarian riddles about an unknown amount of money and its square root.

Already for al-Khwārizmī, however, the *root* had become the fundamental unknown, and the *census* simply the outcome when multiplied by itself.

Al-Khwārizmī offers numerical rules for solving these cases (equation types), but afterwards gives geometric proofs. As I mentioned in my first lecture, these were inspired by the surveyors' riddles, which were still around.

One thing to be observed is that the coefficients are to be either integers or rational numbers. Not only al-Khwārizmī but also Fibonacci and every algebraist until the 16th century avoided irrational coefficients.

The existence of this canon has been established by Jeffrey Oaks.

Except, as we shall see in a moment, apparently in the lost source for some of Fibonacci's problems.

It is an old observation that some of Fibonacci's problems share the mathematical structure with problems found in al-Khwārizmī's algebra, at times also the numerical parameters;

similarly, some problems (with overlap with the former group) relate in one of these or both ways to Abū Kāmil's algebra.

Some are also related to what can be found in al-Karajī's *Fakhrī*.

This overlap already indicates that problems circulated widely, and that shared problems, even with shared parameters, do not prove any direct relationship between two texts.

As I have already said, analysis of the *Liber abbaci* indicates that Fibonacci had an outspoken tendency to be faithful to his sources.

We may therefore conclude that problems whose structure Fibonacci shares with either al-Khwārizmī or Abū Kāmil without sharing the parameters or the procedure can hardly have been borrowed directly from these predecessors.

In order to go beyond this negative conclusion we need to observe that Fibonacci's collection of problems contains a number of closed clusters of problems that must have been adopted together.

The first eleven problems constitute an obvious cluster.

Nine of them share the mathematical structure of a problem from al-Khwārizmī's algebra:

Five come from his list of six illustrations of the basic cases; four come from his collection of "various problems".

Internally in each of these groups, they follow al-Khwārizmī's order; but the two groups are mixed up.

Of the nine that have the same mathematical structure, only two share al-Khwārizmī's numerical parameters.

Only one has the same initial formulation as Gerard of Cremona's translation of al-Khwārizmī, which however is so simple that the coincidence might well be accidental;

> in that case, moreover, the numerical parameters differ, and so do the procedures.

Once again, given his faithfulness when he copies, Fibonacci cannot have used al-Khwārizmī's *Algebra* (in Gerard's or any other version) *directly* for this sequence.

But there can be no doubt that he drew on an introductory work descending from that model, compiled by a writer who was less faithful than Fibonacci when cherry-picking from *his* model.

A transformed problem from Abū Kāmil

A "divided ten" problem not belolnging to a recognizable cluster illustrates Fibonacci's relationship to Abū Kāmil.

It can be expressed in letter formalism:

$$10 = a+b \ , \quad (a/b+10)\cdot(b/a+10) = 122\,{}^{2}/_{3} \ .$$

Abū Kāmil solves the same problem;

al-Karajī instead gives the sum as $143\,{}^{1}/_{2}$.

Al-Karajī posits a to be a *thing;*

a simple transformation then reduces the problem to “*census* plus 16 made equal to 10 roots”, one of the standard cases.

Abū Kāmil posits $^{a}/_{b}$ to be a “large *thing*” (presupposing $a > b$), and $^{b}/_{a}$ to be a “small thing”.

Then, if R stands for the “large *thing*” and ρ for the “small *thing*“,

$$(R+10)\cdot(\rho+10) = 122^{2}/_{3} \ ;$$

since $R\rho = 1$ we thus have

$$1+10\cdot(R+\rho)+100 = 122^{2}/_{3} \ ,$$

whence

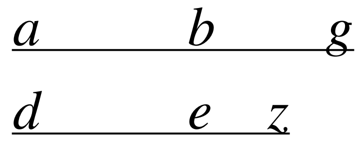
$$R+\rho = 2^{1}/_{6} \ .$$

That is, the problem is reduced to

$$10 = a+b \ , \ \ ^{a}/_{b}+^{b}/_{a} = 2^{1}/_{6} \ ,$$

which Abū Kāmil has already dealt with.

Fibonacci uses a line diagram, lettered *a-b-g-d-e-z*. Here, $ab = de = 10$, while $bg = {}^{a}/_{b}$, $ez = {}^{b}/_{a}$.



Abū Kāmil’s two algebraic unknowns are thus replaced by line segments. The procedure is parallel to that of Abū Kāmil.

Fibonacci’s procedure also leads to the same reference to what has already been dealt with – actually however a reference to what has been dealt with by Fibonacci’s source!

Fibonacci himself has treated the case where the sum of the two fractions is $3^{1}/_{3}$, not $2^{1}/_{6}$.

A clear indication of copying – not directly from Abū Kāmil, however, but at most (in view of the shared structure of the argument almost certainly) from a source building on but reformulating Abū Kāmil’s solution.

Al-Karajī offers a regular *al-jabr* solution.

Abū Kāmil's reduction makes use of a technique that escews reference to the key concepts of *al-jabr*

> (the problem to which he reduces the present one is then solved by means of that technique).

Fibonacci, and his source, also remove anything that could make one think of *al-jabr* techniques (with the same proviso).

The *avere* cluster

The previous example shows Fibonacci's use of an intermediate source but nothing about where this source was produced.

Such information can be derived instead from a cluster that is characterized by using in all problems that ask for a single unknown number the term *avere* for this unknown.

The term is borrowed from some Romance vernacular and means "possession".

It is obviously a loan translation of *māl*.

This term is used nowhere else in the *Liber abbaci*, and it is never used about the algebraic “second power”, which even within this sequence is always *census*.

> Sometimes, indeed, the *avere* is posited afterwards to be a *census*, sometimes to be a *thing*.

The *avere* must already have been present in Fibonacci’s source.

There is indeed no reason that Fibonacci should suddenly on his own choose a new translation – earlier problems as well as the very last one use the standard translation *census* for *māl* in both roles.

Nor is there any reason that elsewhere but not here he should regularly replace an original Arabic initial (non-algebraic) *māl* by *numerus* or *quantitas*.

We cannot exclude that his source was already written in a Romance (but almost certainly not Italian) vernacular.

Much more plausible, however, is a Latin translation prepared in a Romance-speaking environment which borrows terms from its local vernacular

(Catalan, Provençal or Castilian)

The original from which this translation was made is most likely to have been created in al-Andalus.

As we know from correct references to Abū Kāmil's algebra in the *Liber mahameleth*, Abū Kāmil's work circulated there.

Almost half of the problems in the cluster have a close counterpart in Abū Kāmil's algebra.

Moreover, their order is as in that work. Enough to show that the compiler of the original drew on Abū Kāmil's text (perhaps indirectly),

but they are also also sufficiently few and scattered to prove that this was really an independent treatise and no mere redaction.

We may look at a random but characteristic example.

First how Abū Kāmil presents it:

> If it is said, a *māl* of which the two roots and the root of its half and the root of its third are equal to it, how much is this *māl?*
>
> One solves it like this: posit your *māl* to be a *māl*. Then you say, two roots and the root of half of the *māl* and the root of a third of the *māl* are equal to a *māl*.
>
> The *thing* is thus equal to two and root of one half and root of one third, which is the root of the *māl*, and the *māl* is 4 and a half and a third and root of eight and root of five and a third and root of two-thirds.

In symbolic translation – for convenience using x^2 for the *māl* and x for its root:

$$2x+\sqrt{\frac{x^2}{2}}+\sqrt{\frac{x^2}{3}}=x^2\ ,$$

which it is obvious for us to transform into

$$\left(2+\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}\right)x=x^2\ ,$$

a clear case of "roots made equal to *census*", whose rule just as our symbolic solution leads to

$$x=\left(2+\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}\right)\ .$$

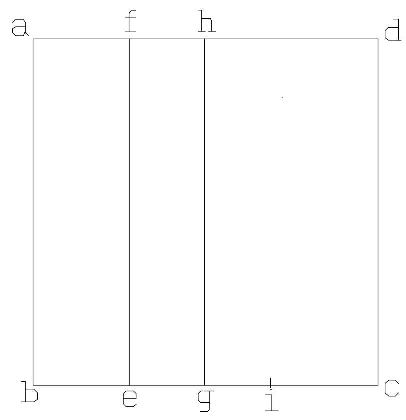
As we can see from Abū Kāmil's text he understands this to perfection.

He is not allowed to say it, however: as I pointed out, irrational "coefficients" were unacceptable.

Therefore Abū Kāmil jumps directly to the result.

The *Liber abbaci* version of the problem evades the difficulty by offering a geometric proof:

> There is an *avere*, of which 2 roots and the root of its half and the root of its third are equal to it.
>
> Posit for this *avere* a *census*; and because two things and the root of the half of a *census* and the root of the third of a *census* are made equal to the *census*, make of the above-written square *ac* a *census*, and two roots of the same *census* will be the surface *dg*, and let the root of the half of the *census* be the surface *eh*, and the root of the third of the *census* the surface *bf*.



Therefore *cg* will be 2 and *eg* will be the root of $^1/_2$ dragma, and *be* will be the root of the third of a dragma. And thus the whole *bc*, which is a *thing*, will be 2 and root of $^1/_2$ and root of $^1/_3$. Therefore multiply this in itself, and $4^5/_6$ and root of 8 and root of $5^1/_3$ and root of $^2/_3$ of one dragma results for the *census*, that is, the *avere* that was asked for.

As we see, Fibonacci considers it in need of no explanation that an area $\sqrt{(^1/_2 \textit{ census})}$ applied to a line of length 1 *thing* produces a breadth $\sqrt{^1/_2}$ (and similarly for $^1/_3$);

that is, the tacit knowledge used by Abū Kāmil serves even here, no new theoretical insight intervenes.

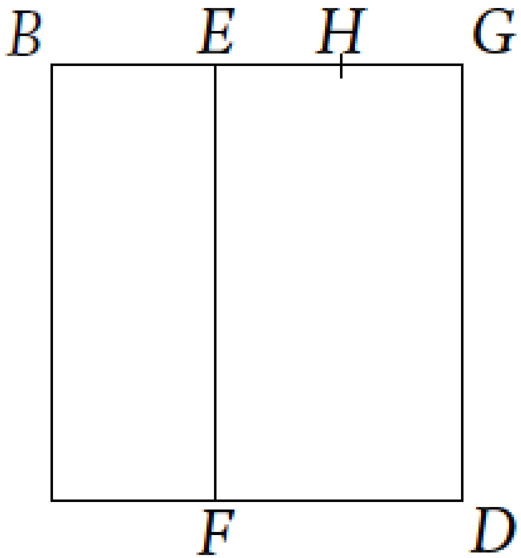
Further we notice the lettering *a-b-c-*....

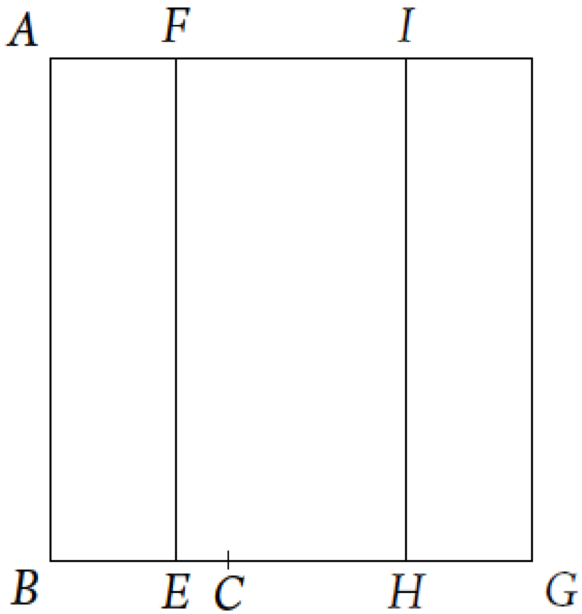
There are a few diagrams lettered *a-b-g-*... within the *avere* cluster; but the large majority are of type *a-b-c-*..., in contrast to what we find in the preceding sections of the problem collection.

So, even though the appearance of the term *avere* shows that Fibonacci drew on an existing Latin (or possibly Romance-vernacular) reinterpretation of Abū Kāmil's algebra, he seems to have intervened himself.

Inspection of the very first problem of the cluster shows us what has happened.

The problem coming just before the first *avere* problem is based on a diagram lettered *b-g-d-e-f-h* (*a* is absent because the corresponding corner of the square is not spoken of and therefore not named at least in the *Liber abbaci* diagram).





Fibonacci borrows from that problem a correct but redundant reference to the classes of Euclidean binomials, which are only mentioned in these two problems and in the one coming just before.

Together with the idea to refer to the Euclidean classes he appears to have taken over the diagram but then adapted it to his own purpose

– namely to solve a problem that he is not allowed to speak of as

$$(10+\sqrt{30})\textit{things}+20 = \textit{census}$$

(now lettered *a*-*b*-*e*-*f*-*h*-*i*-*c* – this time *d* is left out because the corresponding corner is not spoken about).

Personal intervention also seems to be indicated by a number of phrases similar to those that are used elsewhere when an extra explanation or a supplementary proof are provided.

We cannot know whether already Fibonacci's source for the *avere* cluster used geometry to evade the tabooed irrational coefficients or had decided not to respect the taboo.

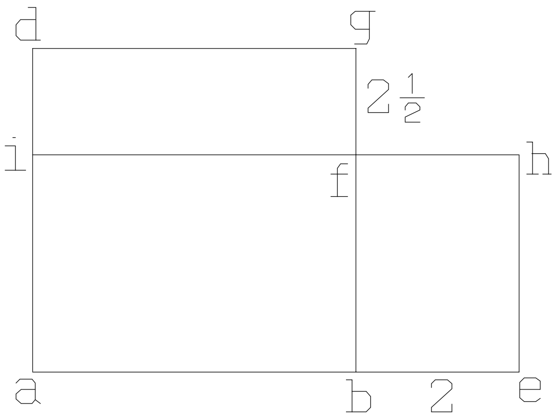
One hint speaks in favour of the latter possibility: In the very first *avere* problem Fibonacci mentions in the beginning the tabooed rule which it falls under, but then does not return to it after his geometric solution.

The reference to the tabooed rule is likely to have been in Fibonacci source. In any case, Fibonacci respects the taboo in what follows.

The money-sharing cluster

A last cluster of interest (also linked to Abū Kāmil) contains problems about a constant or varied amount of money shared between different numbers of men.

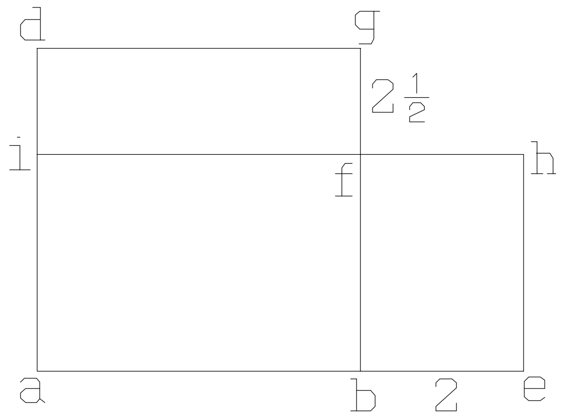
The first of them runs:



I divided 60 between some men, and something resulted for each; and I added two men above them, and between all these I divided 60, and for each resulted $2^1/_2$ less than resulted at first.

Let the number of the first men be the line *ab*, and on it is erected at a right angle the line *bg*, which should be that which falls to each of them of the mentioned 60 *denarii*, and draw the line *gd* equal and parallel to the line *ba*, and the straight line *da* is connected.

Then the space of the quadrangle *abgd* will be 60, when it is connected [*colligatur*] by *ab* in *gb*. [Now, proportion theory is used; it leads to] Thus *fb* contains once and one fourth the number *ba*.



So, posit for the number *ab* a *thing*. *bf* will thus be $1^1/_4$ *thing;* and multiply *ab* in *bf*, and $1^1/_4$ *census* results for the surface *bi* [...].

There are no problems similar to this in the original text of Al-Khwārizmī algebra as reflected in Gerard of Cremona's translation.

The type is equally absent from Robert of Chester's somewhat expanded version.

In extant Arabic manuscripts (from 1222 and later) we have a variant where the amount to be distributed is 1 dirham and one man is added, which results in a difference of $^1/_6$.

This can have crept into the tradition at any moment before 1222; there is no reason to believe it inspired Fibonacci, neither directly nor indirectly.

Instead, Fibonacci's problem is close to a similar one proposed by Abū Kāmil.

Here, 50 dirham are shared, first among some men, them among 3 more, the difference between what each one gets in the two situations being $3^3/_4$ dirham.

The solution follows the same pattern as that of Fibonacci, but instead of using proportions the argument about the diagram is arithmetical all the way through.

In the next problem in the *Liber abbaci*, first 20 is divided between some number of men, next 30 between 3 more; the difference between the shares in the two situations is 4.

Here, a slightly more complicated diagram is used, lettered *a-b-g-d-e-*...; proportion techniques are used again, and followed by an algebraic solution of the resulting equation.

Abū Kāmil offers four problem of the same structure, presenting solutions based on kindred diagrams and never referring to proportions.

The following problems in the *Liber abbaci* have the same mathematical structure.

The solutions, however, are based on diagrams lettered *a-b-c-d-e-...*;

the algebraic *thing* and *census* enter directly in the discussion of the diagrams, while proportions go unmentioned.

The changing lettering of the diagrams suggest that the solutions of the first two problems build on a source that ultimately goes back to Abū Kāmil

while the others are of Fibonacci's own brew.

The former reformulate Abū Kāmil's solutions in proportion terms; Fibonacci, in his own (more straightforward) solutions, does not mention them.

In the last problem from the sequence, first 10 are divided between a certain number of men, next 40 between 6 more; they get the same in the two cases.

Thinking in terms of proportions would lead to $\frac{h+6}{h} = \frac{40}{10}$, and thus $\frac{6}{h} = \frac{40-10}{10}$, and finally $6 \cdot 10 = 30 \cdot h$. But Fibonacci, again apparently working on his own, has no such preferences on the present occasion.

He observes, without any appeal to algebra, that the 30 extra monetary units must be the share of the 6 extra men, each of whom therefore gets 5. Since the first men get the same, their number must be $10 \div 5 = 2$.

There can be no doubt that this sequence is part of a cluster adopted from a single source, which on its part seems to have been inspired by Abū Kāmil

(shared inspiration cannot be totally excluded, but the agreement in order speaks against this possibility).

For the last three problems, however, Fibonacci seems to have presented a simpler solution of his own making.

There is no direct evidence for where the original was produced from which Fibonacci took this cluster, not even for whether he used an Arabic text or a Latin translation.

At most we can say that the predilection for proportion techniques may make us think of the *Liber mahameleth*, which again would point to al-Andalus.

We can thus discern two distinct clusters in Fibonacci's collection of algebraic problems, well apart in his text, both of which almost certainly draw on Abū Kāmil's algebra but do so indirectly.

Clear differences in mathematical style indicate that the two clusters derive from different intermediaries. For one of them at least Fibonacci relies on a Latin translation.

Jacopo's quasi-algebra

To finish, we shall have to look at a vernacular source from 1307.

After the algebra section in Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307, follows four problems which in our perspective are algebraic of the second and third degree but were not seen like that by Jacopo.

They all deal with the wages of the manager of a *fondaco*, a warehouse located abroad, supposed tacitly to increase geometrically over three or four years.

> That this is tacitly supposed shows that we are confronted with what was considered a standard problem at the location where the problems originated.

Designating the salaries a, b, d and e (e only in (2) and (4)), we may express the problems thus:

(1) $a+d = 20$, and $b = 8$.

(2) $a = 15$, $e = 60$.

(3) $a+e = 90$, $b+d = 60$.

(4) $a+d = 20$, $b+e = 30$.

In all cases, $a :: b :: d :: e$ (the last step again only in (2) and (4)).

It is thus clear that $ad = b^2$, $ae = bd$.

All solutions are given as numerical prescriptions only, there is neither *al-jabr* algebra nor line diagrams. It is easy, however, to change the prescriptions unambiguously into formulae.

For (1), $a+d = 20$, and $b = 8$, this solution is given:

$$a = \frac{a+d}{2} - \sqrt{(\frac{a+d}{2})^2 - ad} \quad \text{and} \quad d = \frac{a+d}{2} + \sqrt{(\frac{a+d}{2})^2 - ad} \; .$$

It follows from $ad = b^2$ and application of *Elements* II.5 in “key” version.

Problem (2) , $a = 15$, $e = 60$, is solved as

$$b = \sqrt[3]{\frac{d}{a} \cdot a^3} \ , \ \ d = \sqrt[3]{(\frac{d}{a})^2 \cdot a^3} \ ,$$

which asks for nothing but familiarity with the theory of continued proportions (duly generalized so as to accept irrational radicals) – namely the finding of two mean proportionals.

If we think in terms of a factor of proportionality *s*, as we know it from the *Liber mahameleth*, it can be solved as the algebraic problem "cubes equal number," for which Jacopo has presented the rule earlier on

– but no such link is made, and it is doubtful whether Jacopo really understood algebra beyond the second degree (he offers no examples, only rules).

Problem (3) , $a+e = 90$, $b+d = 60$, is more astonishing.

It is indeed an irreducible third-degree problem, solved by means of the formula

$$a{\cdot}e = b{\cdot}d = \frac{(b+d)^3}{3(b+d)+(a+e)} ,$$

which is easily justified if we think in terms of a factor of proportionality s (well known in the *Liber mahameleth)*. Then, indeed,

$$\frac{(b+d)^3}{3(b+d)+(a+e)} = \frac{a^3s^3(1+s)^3}{a\,(3s+3s^2+1+s^3)} = a^2s^3 = a{\cdot}as^3 = as{\cdot}as^2 .$$

How it had been found is quite another matter.

In any case, once $a{\cdot}e = b{\cdot}d$ are found, $a+e$ and $b+d$ being already given, all four can be found (and are indeed found) by means of *Elements* II.5 in “key” version.

Without identifying it, problem (4) , $a+d = 20$, $b+e = 30$

finds the factor of proportionality s as $(b+e)/(a+d)$, whence

$$a = \frac{a+d}{1+p^2}\ ,\ \ d = (a+d)-a\ ,\ \ b = \frac{b+e}{1+p^2}\ ,\ \ e = (b+e)-b\ .$$

Since the factor of proportionality is actually found and used here, we may be confident that it also underlies the solutions of problems (2) and (3).

Already this, together with the tacit use of *Elements* II.5 in “key version”, suggests kinship with *Liber mahameleth* and *Liber abbaci*, chapter 15 part 1.

The idea to use a (pseudo-)practical problem as a starting point or pretext for deeper study, on the other hand, recalls what was done to the unknown heritage and the treatment of commercial problems in the *Liber mahameleth*.

Scattered later sources until Pacioli, Tartaglia and Cardano do contain some similar problems, with variations that suggest Jacopo not to have been the only channel through whom the problem type reached the Italian abbacus environment.

But there is nothing similar in earlier known sources.

We may therefore assume with fair certainty that even this group of problems was drawn from the same al-Andalus reservoir as the three Latin sources that we have looked at.

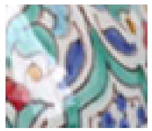
All in all:

Once upon a time there was a beautiful vase.

It was shattered by crusade and fratricide wars.

Most of the sherds were brushed away in the course of centuries.

Only the few pieces that by accident had been inserted into other structure were saved.

Mostly these pieces have gone unnoticed, but they *can* be brought to light.

Shared chemistry suggests that they really belong together.

All that, however, gives nothing but a very vague idea of the original vessel.