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Argument and demonstration in particular in Greek Antiquity

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My central topic today will be ancient Greek argued geometry and the gradual emergence of axiomatization as a practice and as an ideal.

In the end I shall offer some sketchy perspectives.

At first, however, I shall introduce and exemplify two concepts that will serve.

The "locally obvious"

How do we explain a point of mathematics to somebody outside an ordinary context of teaching:

We appeal to what we suppose the actual interlocutor can be supposed to understand or accept as true.

(Not only when we explain mathematics, of course)

This is what I shall call the *locally obvious* – the alternatives "heuristic" or "intuitive argument" being both broader while omitting the actual context and audience.

I shall illustrate this with an example from Dardi of Pisa's *Aliabraa argibra*, written in 1344, probably in Venice.

Its first part presents the arithmetic of monomials, binomials and polynomials containing radicals.

One passage explains how to divide *number* by *number plus square root,* with the example $\frac{8}{3+\sqrt{4}}$.

(I transcribe in modern notation – Dardi has *più* where I write "+", *meno* where I have "–", and ℞ where I use √. Finally, I write the division as a fraction)

Dardi was an “abbacus teacher”, that is, a teacher in an abbacus school, a school frequented by merchants and artisan sons for 1½–2 years around age 11–12,

teaching basic applied and commercial arithmetic

from Hindu-Arabic numerals and their use and the rule of three, to composite interest, single false position, alloying, and the like.

Algebra, in particular Dardi’s algebra, goes far beyond this. But he is no university scholar, and has no Euclidean training.

Building on what he has already taught, Dardi calculates that $(3+\sqrt{4})\cdot(3-\sqrt{4}) = 3^2-(\sqrt{4})^2 = 5$.

As he explains, he uses rational roots "as if they were surds", which allows control.

Therefore he knows that

$$\frac{5}{3-\sqrt{4}} = 3+\sqrt{4}$$

and that

$$\frac{5}{3+\sqrt{4}} = 3-\sqrt{4}\ .$$

What we need to find is

$$\frac{8}{3+\sqrt{4}}\ .$$

So far, nothing seems amazing from our point of view.

But now comes something unexpected. Dardi makes appeal to the rule of three, which tells him that

$$\frac{8}{3+\sqrt{4}} = (8\cdot[3-\sqrt{4}])\div 5 = (24-\sqrt{256})\div 5$$

which he then in agreement with abbacus algebra aesthetics reduces to

$$4\frac{4}{5} - \sqrt{10\frac{6}{25}} \ .$$

What precisely was the rule of three for Dardi?

Not the *problem* to find an unknown q (or p) from "if q is to p as Q to P" (where p and q may stand for "quantity" and "price", respectively),

nor for *whatever method* can be used to solve that problem.

The rule of three is the specific method which first multiplies and then divides, and only that.

In the Italian abbacus school environment it was taught in words like these:

> If some computation was said to us in which three things are proposed, then we shall multiply the thing that we want to know with the one which is not of the same (kind), and divide in the other.

repeated more or less verbatim in almost all abbacus writings that formulate the rule.

This is thus certainly what Dardi referred to.

The rule was taught unexplained; it is indeed difficult to explain, since the intermediate product has no concrete meaning.

The recourse to the rule of three was certainly meant by Dardi as an explanation. Other abbacus writers use it in a similar way.

Is it a demonstration? Probably even Dardi did not think of it in terms like that, but rather as what we might express as a “reasoned procedure”.

We may compare with the way we ourselves may have been taught to perform the same kind of division – I myself around the age of 14.

We would have been told to multiply the numerator and the denominator of $\frac{8}{3+\sqrt{4}}$ by $3-\sqrt{4}$,

$$\frac{8}{3+\sqrt{4}} = \frac{8\cdot(3-\sqrt{4})}{(3+\sqrt{4})\cdot(3-\sqrt{4})} = \frac{8\cdot(3-\sqrt{4})}{3^2-(\sqrt{4})^3} = \frac{24-8\sqrt{4}}{9-4} = \frac{24-8\sqrt{4}}{5}\,.$$

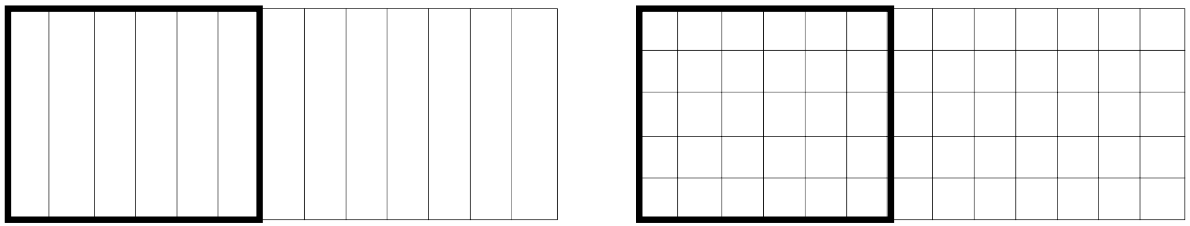
Even this is a reasoned procedure, and we might spontaneously tend to see it as more akin to demonstration.

But how did we know that a fraction does not change its value when numerator and denominator are multiplied by the same number?

And would $3 - \sqrt{4}$ be a number in the sense corresponding to the argument behind this manipulation?

> We remember that Dardi's roots, though rational, should be understood as surds.

It certainly was not. At an earlier moment we may have been presented with an explanation of the expansion of, say, $\dfrac{6}{13}$ into $\dfrac{5\cdot 6}{5\cdot 13}$ corresponding to this diagram:



Left, in heavy outline, $\frac{6}{13}$ of a rectangle – 6 out of 13 equal strips. Right the same, now $5\cdot 6$ out of $5\cdot 13$ equal squares, that is, $\dfrac{5\cdot 6}{5\cdot 13}$ of the rectangle.

To make that a rigorously valid argument in the case of irrational factors would require something like an Archimedean exhaustion, and thus also explicitly stated second-order logic.

In any case, when we were confronted with $\frac{8}{3+\sqrt{4}}$ we had long forgotten the argument for the possibility of reduction or expansion of fractions (*if* we had ever been presented with one);

we had just got accustomed – just as Dardi's model reader was accustomed to the use of the rule of three.

Critique

What is obvious for one person (for instance, the teacher) may not be obvious to another one (for instance, the student); and what at first seems obvious may even become doubtful for the same person at second thoughts.

That is where *critique* sets in, reflections about *Möglichkeit und Grenzen*, "possibility and limits", in Kant's words from the opening of the Third Critique.

This is another concept I shall refer to.

I shall illustrate this with an Old Babylonian example – the text YBC 6967, from somewhere between 1750 BCE and 1600 BCE, already mentioned in my previous lecture.

It deals with two numbers, *igûm and igibûm*, whose product is known to be 60; moreover, the *igibûm* exceeds the *igûm* by 7.

Since the use of the place value system is immaterial for our present purpose, I use our number notation.

The *igibûm* over the *igûm*, 7 it goes beyond
igûm and *igibûm* what?
You, 7 which the *igibûm*
over the *igûm* goes beyond
to two break: $3^\circ\frac{1}{2}$;
$3^\circ\frac{1}{2}$ together with $3^\circ\frac{1}{2}$
make hold: $12^\circ\frac{1}{4}$.
To $12^\circ\frac{1}{4}$ which comes up for you
1` the surface join: $1`12^\circ\frac{1}{4}$.
The equal of $1`12^\circ\frac{1}{4}$ what? $8^\circ\frac{1}{2}$.
$8^\circ\frac{1}{2}$ and $8^\circ\frac{1}{2}$, its counterpart, lay down.
$3^\circ\frac{1}{2}$, the made-hold,
from one tear out,
to one join.
The first is 12, the second is 5.
12 is the *igibûm*, 5 is the *igûm*.

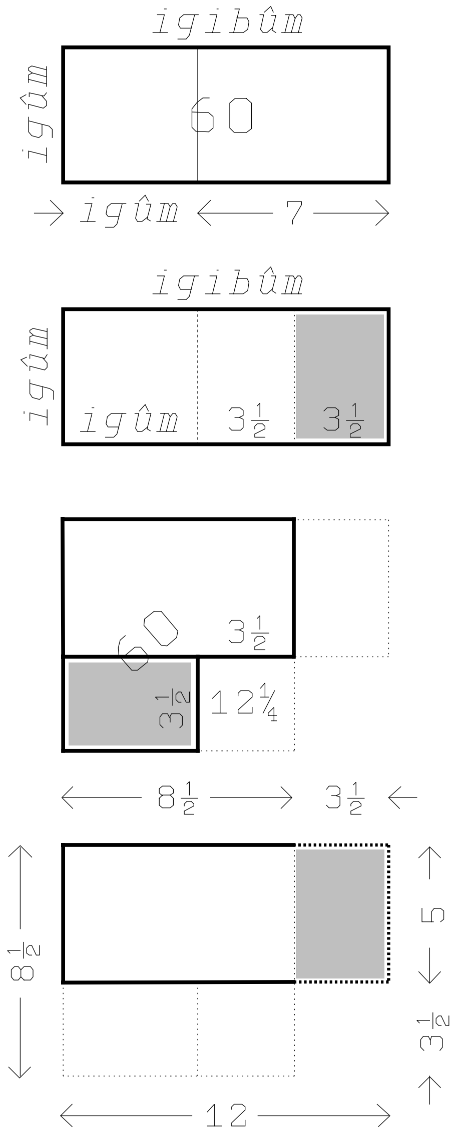
The *igibûm* and the *igûm* are represented by the length and the width of a rectangle.

The excess of the length over the with is broken into two, and the outer half moved around so as to form with the rest a gnomon, still with area 60.

The side of the missing square is $3\frac{1}{2}$, and its area thus $12\frac{1}{4}$.

The completed square thus has an area $60+12\frac{1}{4}$ and side $8\frac{1}{2}$. From this it would be easy to find the *igibûm* as $8\frac{1}{2}+3\frac{1}{2} = 12$ and the *igûm* as $8\frac{1}{2}-3\frac{1}{2} = 5$.

That is also what earlier texts dealing with analogous configuration do, using the contracted formulation "join and tear out".



A

B

C

D

It is further reflected in the reversal of the order in the last two lines, 12 resulting from addition, 5 from subtraction.

In the preceding lines, however, subtraction precedes addition, against normal Babylonian habits. The reason is regard for concrete meaningfulness: we cannot put something back in place before it is made available.

This cannot be evidence of “a primitive mind not yet prepared for abstraction”, as has been supposed, since earlier texts analogous texts are less “primitive”, prescribing solely “join and tear out”.

Instead, at some moment, some teacher, perhaps challenged by a student, perhaps as a result of his own second thoughts, has discovered that the inherited way of speaking is deprived of concrete meaning; that is, he has engaged in *critique*.

Critique is not a conspicuous characteristic of Old Babylonian mathematical texts. I know of only one other instance, which I shall not present here.

We should not wonder: Old Babylonian was basically taught as a means for administration.

Even though it created a higher level of "supra-utilitarian" problems, the norms governing the practice out of which these grew asked for finding "the right number", not for theoretical justification beyond what might be didactically useful.

For pedagogical reasons, it might offer explanations based on the locally obvious. Critique played a peripheral role only.

Demonstration, critique, and the culture of liberal arts

From Classical Greco-Roman Antiquity comes the concept and ideal of "liberal arts", knowledge bodies that have no technical use but are considered goals in themselves.

We should take note that the famous "cycle" of seven Liberal Arts (grammar, rhetoric, dialectic; arithmetic, geometry, astronomy, harmonics) was only formed during or after Plato's mature years,

and that the supposed "seven" were normally two and nothing more – grammar, that is, good and correct use of language, and rhetoric.

Things have to be reduced to due proportions.

Yet there *were*, as we know, people engaged in “liberal” mathematics during Classical Antiquity – and not only Euclid, Archimedes and Apollonios.

According to Reviel Netz’s estimate from 1999 of the number of those who at some moment in life made a piece of explicitly reasoned mathematics, 144 have left at least minimal direct or indirect traces;

perhaps some 300 were still known by name in Late Antiquity

– and in total perhaps 1000, one born on the average per year,

but certainly with a more uneven distribution than simple randomness would suggest (and quite possibly considerably fewer).

Their appearance also precedes the formation of the *cycle* of Liberal Arts.

It *almost* had to – how could the quadrivial arts (arithmetic, geometry, astronomy and harmonics) become part of the cycle if they did not already exist?

Yet we should beware that what entered the cycle were, at least by name, Pythagorean fields of interest;

to which extent these corresponded to the reasoned theoretical fields we know from Aristotle's time onward is unclear.

Even the nature of the mathematics which according to Plato should be taught to the guardians of his republic is subject to doubt.

There is no evidence (not even suggestive) that his "arithmetic" was something like the theory of *Elements* VII–IX – after all, the word basically means "counting", and how far Plato stretched this is not clear from his text.

A passage in Aristotle's *Metaphysics* concerns, not the state of affairs at the moment when the *Republic* dialogue is supposed to have taken place but Plato's own teaching at the moment when Aristotle was working at the Academy or later).

After other objections against Plato's identification of numbers with ideas it is pointed out that

> not even is any theorem true of them, unless we want to change the objects of mathematics and invent doctrines of our own.

That is: whatever Plato maintains in his mature philosophy about number has nothing to do with the theoretical arithmetic that had been created no later than the fourth century BCE.

On the other hand we know that some kind of theoretical mathematics existed during the second half of the fifth century BCE.

Famously, the possibility of *incommensurability* had been discovered by then, most likely by Pythagorean *mathematikoi* (*not* by Pythagoras),

and we know that first Theodoros and later Theaitetos worked on this topic – Plato's dialogue *Theaetetus*, though written after 370 BCE, can be considered testimony.

From the reports about and fragments from Plato's contemporary Archytas we also know about investigation of the three main mathematical *means* (arithmetic, geometric, harmonic).

At least irrationality is beyond what could be of interest in any productive or administrative practice;

a connection between the theory of means and the theory of harmonics can be presupposed,

but then the theory of harmonics was a mathematical theory, and its relation to practised music questionable (questioned indeed by Aristotle's associate Aristoxenos).

The theory of means was also linked to the search for two mean proportionals; even this was of no interest for administrators or master builders.

We are ignorant, however, not only of the precise arguments used by Theodoros and Archytas but also of their overall argumentative style.

Happily, we know somewhat more about Hippocrates of Chios – but what exactly?

Firstly, of course, the name tells us that he came from the island *Chios*.

From remarks made by Aristotle combined with the developing political situation in the Aegean it follows that he was active before 420 BCE, and that a school ("those around Hippocrates") existed in Athens, which was interested in astronomy.

Then, Proclos commentary to *Elements* I states that

> Following [Oinopides of Chios and Anaxagoras] Hippocrates of Chios, who invented the method of squaring lunes, and Theodorus of Cyrene became eminent in geometry. For Hippocrates wrote a book on elements, the first of whom we have any record who did so.

Proclos belongs to the late fifth century CE; he was the last outstanding head of "Plato's" Academy.

His commentary to Euclid is written with an unmistakeable Platonic-Neopythagorean intent, but also draws honestly on works not sharing this orientation.

Proclos also reports a solution to the "Delic problem", the duplation of the cube, which Eratosthenes ascribes to Hippocrates. It shows that what is needed is to find two mean proportionals between 1 and 2 (an instance of the "reduction method").

This quotation and information comes from the “catalogue of geometers” outlining the progress from Thales to Euclid.

It is mainly drawn from Eudemos, very close to Aristotle, who appears on his part to have had a tendency to reinterpret history in an Aristotelian perspective.

The "book on elements" is often supposed to be an early version of "the *Elements*".

Seen in the light of the work on lunes (that is, planes figure contained by two circular arcs), on which in a moment, what Proclos says refers instead to the original meaning of "elements"/στοιχεῖα, the elementary building blocks from which composites are made.

The letters composing a word are thus στοιχεῖa; the word is also used about the "elements" earth, water, air and fire.

This is actually explained by Proclos a little later.

Proclos thus does not claim that Hippocrates presented the world with an early axiomatic system. Analysis of the work on lunes will show us what was probably the character of Hippocrates's "elements".

The lunes

At some moment after 500 CE, Simplicios of Cilicia – Aristotelian and prudent Neoplatonist – wrote a commentary to Aristotle's *Physics*.

One of the points treated is a logical fallacy which Aristotle speaks of in these words:

> We are not bound to answer every kind of objection we may meet, but only such as are erroneously deduced from the accepted principles of the science in question.
>
> Thus, it is the geometer's business to refute the squaring of the circle that proceeds by way of equating the segments, but he need not consider Antiphon's solution.

Since in Aristotle's view it was not made according to the principles of geometry, Antiphon's solution is generally supposed to have been by means of approximations by polygons.

What was the other?

A possible key is offered by Aristotle in *On Sophistical Refutations*,

> Then there are those false reasonings which do not accord with the method of inquiry peculiar to the subject yet seem to accord with the art concerned.

> For false geometrical figures are not contentious (for the resultant fallacies accord with the subject-matter of the art),

> and the same is the case with any false figure illustrating something which is true, for example, Hippocrates' false figure or the squaring of the circle by means of lunes.

If "Hippocrates's false figure" (ψευδογραφέμα) is the same as the "squaring of the circle by means of lunes", then we have an answer to the question raised by the *Physics* passage.

From Simplicios we know about Hippocrates's investigation of the lunes.

It is obviously reasoned – the three "classical problems", one of which (the squaring of the circle) is the inspiring background to Hippocrates's question – only make sense as theoretical problems.

As we shall see, however, there is no trace of axiomatics – the argument makes use of two principal tools, together with some properties of his diagrams which he tacitly takes for granted as (locally) obvious.

In his commentary to Aristotle's *Physics*, Simplicios reports Hippocrates from two sources – Alexander of Aphrodisias's commentary to the same *Physics* passage (written ca 200 CE), and Eudemos

> (that is, the "Eudemos" which Simplicios knows – it is disputed whether he had access to the original text or an abridgement or epitome; hardly decidable and not important for what I try to do here).

Simplicios does not *quote* Alexander, he reports what he writes.

Eudemos he does quote, though expanding with such references to Euclid's theorems that he supposes Hippocrates to have known but believes Eudemos to have left out (and also in other ways).

Even under these conditions it is obvious that the two versions differ, but also that none of them can descend from the other through abbreviation and distortion.

I shall come back to the contents of the two versions (thus substantiating what I just said).

But first: Can Hippocrates really have made two different versions?

He certainly can – several ways can be imagined.

The print world was (and still is) familiar with “first” and “second, revised and augmented” editions;

in the digital world we have the experience of “preliminary versions” which end up in odd corners of the web from where the author has no possibility to remove them.

Neither of these was the situation of Hippocrates; his was the world of incipient manuscript culture, which however in this respect was not too different.

Here, any preliminary version of a treatise lent to somebody might start to circulate on its own as long as somebody found it interesting enough to copy it or have it copied.

Even later in Greco-Roman Antiquity, where booksellers made it possible to buy more or less standardized versions, private copying remained important.

This, in particular, was the case for scientific and philosophical books, where the audience was too restricted to make it worthwhile for a bookseller to keep a master copy.

So, a second "much augmented and revised version" may well have been prepared by Hippocrates himself.

But Hippocrates was also the centre of some kind of “school”.

There is no reason to suspect Hippocrates of having taught in his school a doctrine different *in character* from what went into a published work;

but any teacher who has written a book on the basis of lectures made earlier on will know that the two become different.

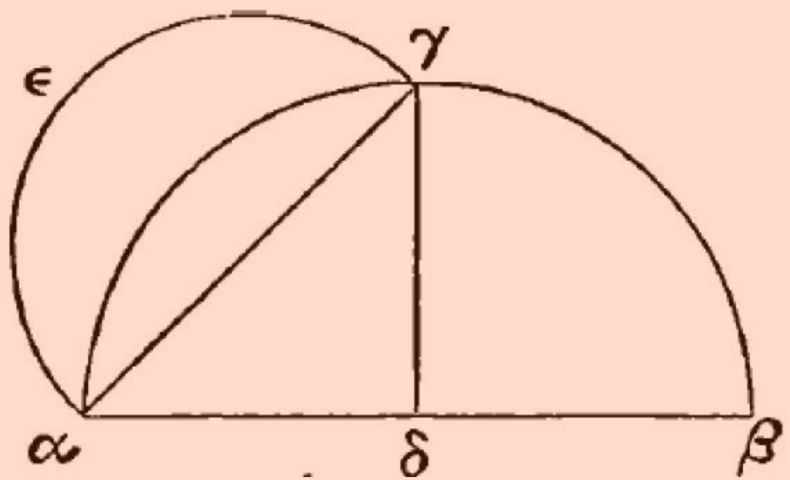
Alexander may report an early version of the investigation of the lunes

(perhaps based on what was written down by listeners, perhaps on notes prepared by Hippocrates himself),

while Eudemos uses a revised and improved version published afterwards.

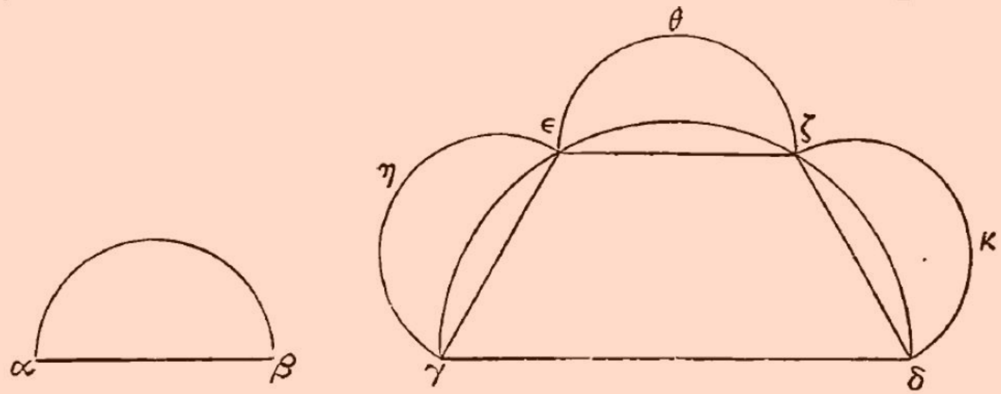
Now back to Simplicios's report of Alexander. I shall not go through it.

A Let a semicircle αβγ be described on the straight line αβ; bisect αβ in δ; from the point δ draw a perpendicular δγ to αβ, and join αγ; this will be the side of the square inscribed in the circle of which αβγ is the semicircle. On αγ



describe the semicircle αεγ. Now since the square on αβ is equal to double the square on αγ (and since the squares on the diameters are to each other as the respective circles or semicircles), the semicircle αγβ is double the semicircle αεγ. The quadrant αγδ is, therefore, equal to the semicircle αεγ. Take away the common segment lying between the circumference αγ and the side of the square; then the remaining lune αεγ will be equal to the triangle αηγδ; but this triangle is equal to a square. Having thus shown that the lune can be squared, Hippocrates next tries, by means of the preceding demonstration, to square the circle thus:—

B Let there be a straight line αβ, and let a semicircle be described on it; take γδ double of αβ, and on it also describe a semicircle; and let the sides of a hexagon, γε, εζ, and ζδ be inscribed in it. On these sides describe the semicircles γηε, εθζ, ζκδ. Then each of these semicircles described on the sides of the hexagon is equal to the semicircle on αβ, for αβ is equal to each

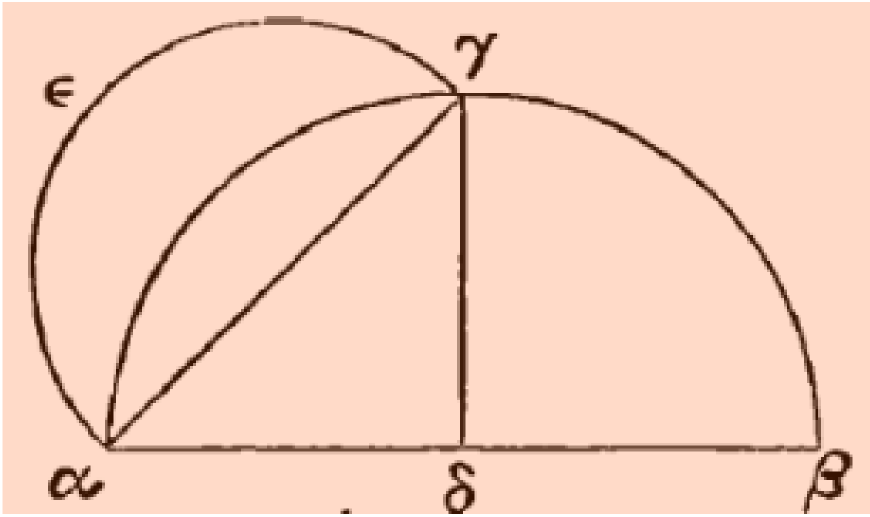


side of the hexagon. The four semicircles are equal to each other, and together are then four times the semicircle on αβ. But the semicircle on γδ is also four times that on αβ. The semicircle on γδ is, therefore, equal to the four semicircles – that on αβ, together with the three semicircles on the sides of the hexagon. Take away from the semicircles on the sides of the hexagon, and from that on γδ, the common segments contained by the sides of the hexagon and the periphery of the semicircle γδ; the remaining lunes γηε, εθζ, and ζκδ, together with the semicircle on αβ, will be equal to the trapezium γε, εζ, ζδ. If we now take away from the trapezium the excess, that is a surface equal to the lunes (for it has been shown that there exists a rectilineal figure equal to a lune), we shall obtain a remainder equal to the semicircle αβ ; we double this rectilineal figure which remains, and construct a square equal to it. That square will be equal to the circle of which αβ is the diameter, and thus the circle has been squared.

The treatment of the problem is indeed ingenious; but the wrong conclusion arises from assuming that as demonstrated generally which is not so; for not every lune has been shown to be squared, but only that which stands over the side of the square inscribed in the circle; but the lunes in question stand over the sides of the inscribed hexagon. The above proof, therefore, which pretends to have squared the circle by means of lunes, is defective, and not conclusive, on account of the false-drawn figure which occurs in it.

In brief:

Section A shows that the lune contained between two semi-circles whose diameters in square are in ratio 1 : 2 is equal to a triangle.



Which are the tools?

(1) that the square on the diagonal of a square is twice the square itself;

(2) that areas of circles are to each other as the squares on the diameters;

(3) what I call the “Pythagorean rule” since nothing like a proof is made appeal to;

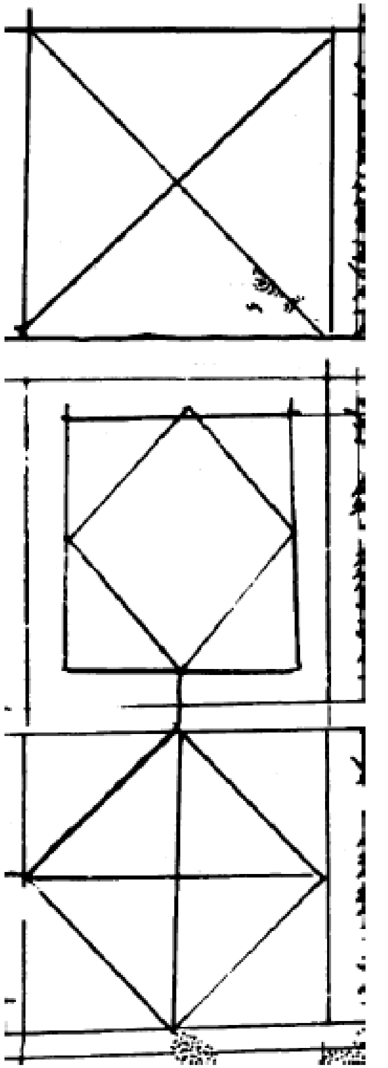
(4) additive and subtractive arithmetic applied to areas.

None of them are supported by arguments; they stand as “what we know but hardly need to state”.

(1) is familiar from Plato's *Menon* – it can be found by anybody playing around with subdivision of squares.

But it is also shown in an Old Babylonian tablet (BM 15285), along with other much more complicated subdivisions.

(2) is a special case of knowledge which already in the late third millennium had been the very basis for Ur III tables of fixed coefficients (staying alive afterwards): The areas of similar figures are proportional to the squares on a characteristic linear dimension.



Two commentaries:

In the Eudemos text it is said that Hippocrates *X*ed this. The verb *X* (δείκνυμι) can be translated "prove", in particular in later mathematics. But it may also simply mean "point out" or "explain".

"Similarity", as defined by Euclid (and already referred to by Aristotle), is a complex derived concept, involving angles and proportions.

But "of similar shape" is direct and immediate, and had been presupposed as such in Mesopotamian measuring geometry since before 2000 BCE.

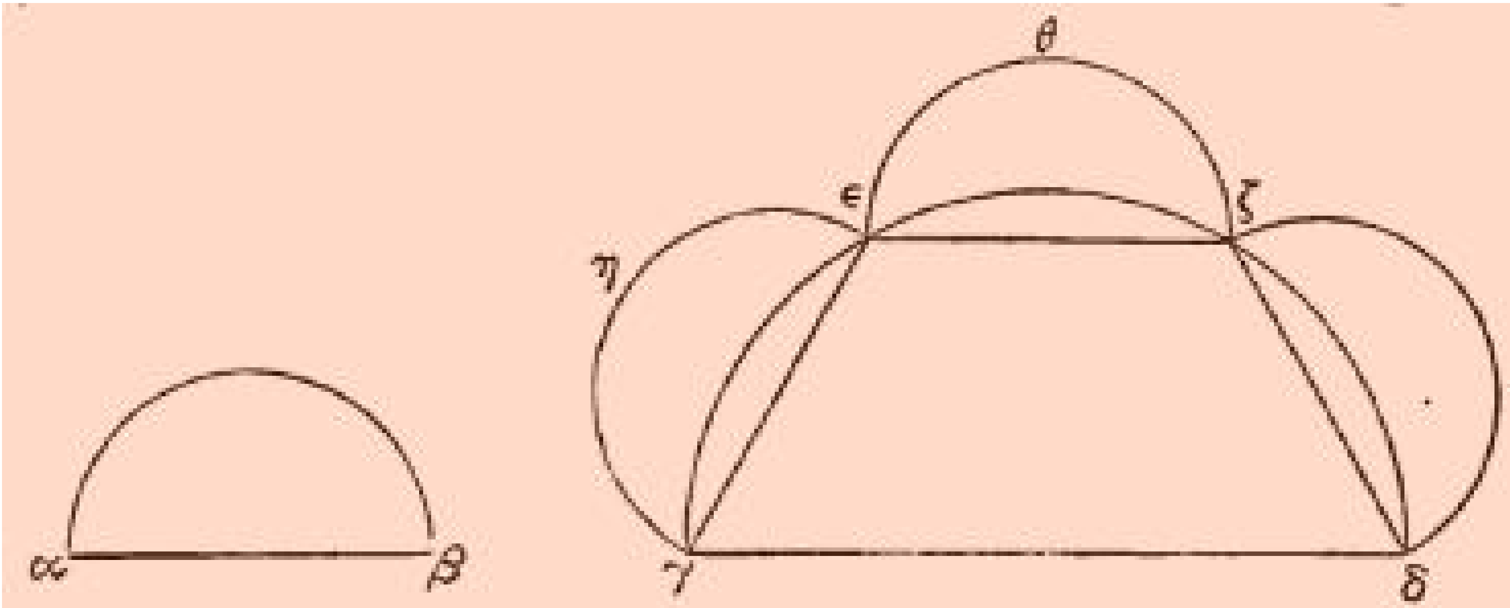
(3) had also been used and known (once quoted in abstract formulation!) in Mesopotamia since the early second millennium BCE.

(4) is a presupposition without which numerical measuring of areas would be meaningless. Any surveyor would take it as evident and would hardly see any need to state it as a principle. (Nobody before Hilbert did, I believe.)

All in all, Hippocrates builds on the "locally obvious", presuppositions which will be accepted as obvious by the real or envisaged audience.

Nothing goes beyond what a Mesopotamian scribe school teacher would have had to tell his students when explaining to them the use of the table of constants.

Section B deals with a more intricate configuration.



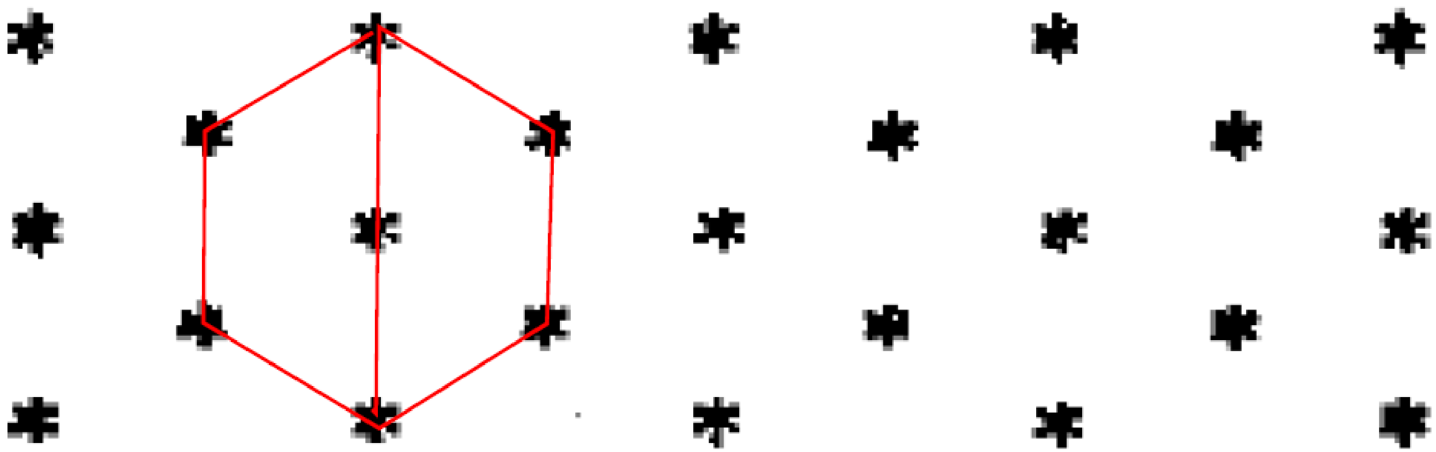
The concluding part of this section finds that the three small lunes together with the semicircle on αβ equals the trapezium γεζδ.

It then claims that since a particular lune has been squared in section A even these lunes can be squared – and therefore also the semicircle.

This is supposed by Alexander to be the fallacy referred to by Aristotle.

Only two new presuppositions are made, the first introduced by a simple “then”: The first is that the side of a regular hexagon equals the semi-diameter of the circumscribed circle.

Even this is old knowledge; it is explicit in a late Old Babylonian mathematical text (*ca* 1600 BCE). The principle is also known from the most efficient way to plant threes in a plantation – in Latin known as *quincunx*, shown like this in Lewis & Short, *Latin Dictionary*, red added):



The second is that a rectilineal figure can be transformed into a square.

This is obviously *Elements* II.14.

Euclid restricts himself to showing the transformation of a rectangle into a square, leaving the preceding steps unexplained.

Euclid's method is to find a mean proportional (not identified as such).

One way to do that (old knowledge once again) would be to transform the rectangle into a gnomon and then to apply the Pythagorean rule.

But Hippocrates most likely knew other ways. This is suggested by his transformation of the doubling of the cube.

Alexander's pinpointing of the fallacy of the squaring of the circle is evidently correct. There is widespread agreement that Hippocrates cannot be responsible for the logical blunder.

In Heath's words, "It is evident that this account does not represent Hippocrates's own argument, for he would not have been capable of committing so obvious an error".

This would not prevent Alexander from being right in identifying what we find here with the error Aristotle refers to: Aristotle may have known the same text as Alexander.

Heath, and other philologists, mistake the perfected mathematics they know from textbooks for the work of creative mathematicians.

- Eminent mathematicians, when working at the extreme limit of their understanding (as Cardano dealing with complex numbers) sometimes err;

- Even Andrew Wiles glorified proof of Fermat's Last Theorem originally contained an error and had to be perfected.

- We may take it for granted that the proof which Fermat believed to have found but which was too large for the margin in his Diophantos as also fallacious.

But if the text used by Alexander stems from a listener, a member of Hippocrates's circle, it could also reflect an over-enthusiastic interpretation of what the master had meant as part of a *search* for a squaring by means of lunes.

That Hippocrates was indeed engaged in this kind of systematic exploration seems to follow from the "Eudemos" version.

There are minor disagreements about how to identify Simplicios's additions to what he found in Eudemos.

One might fear circularity in the argument: By removing what does not fit our ideas about how and what Eudemos would write, are we not merely confirming our preconceived ideas?

One observation made by Oskar Becker eliminates this danger in most cases:

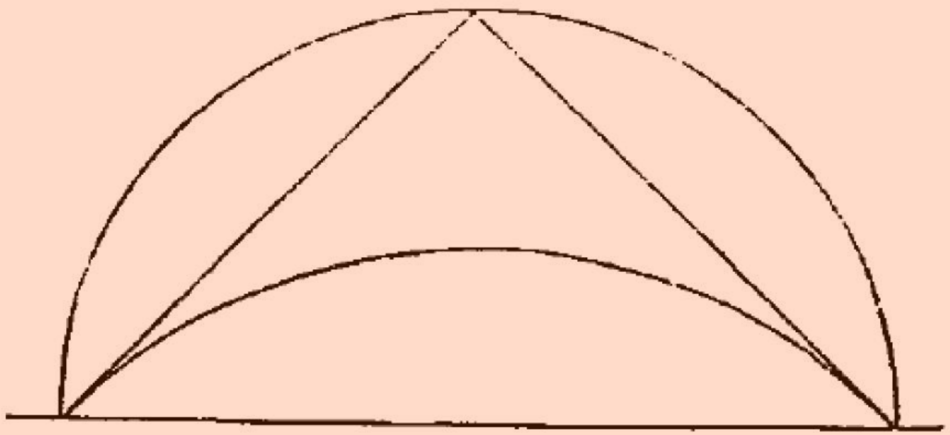
> passages that indubitably come from Eudemos speak (e.g.) of "the line on AB", as Aristotle would do; text which indubitably comes from Simplicios's hand speaks of "the line AB".

Here I copy Ivor Bulmer-Thomas translation. Once again, I shall not go through it.

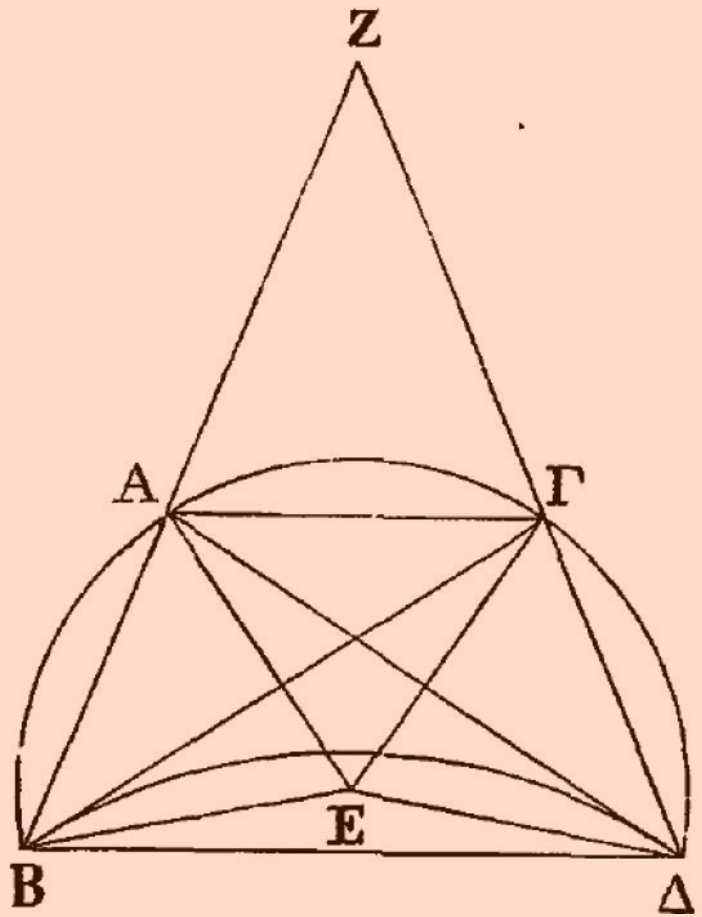
[Eudemos] writes thus in the second book of the *History of Geometry*.

C The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems useful [should be “first of what was useful ...”] to his purpose, that similar segments of circles have the same ratios as the squares on their bases. And this he proved by showing [literally “explained by pointing out” – both verb are δείκνυμι] that the squares on the diameters have the same ratios as the circles.

D Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the Sides.[Here Thomas leaves out a passage which Ferdinand Rudio considers Eudemian, and which in view of vacillating terminology is even likely to go back to Hippocrates]. Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared. In this way, taking a semicircle as the outer circumference of the Lune, Hippocrates readily squared the lune.

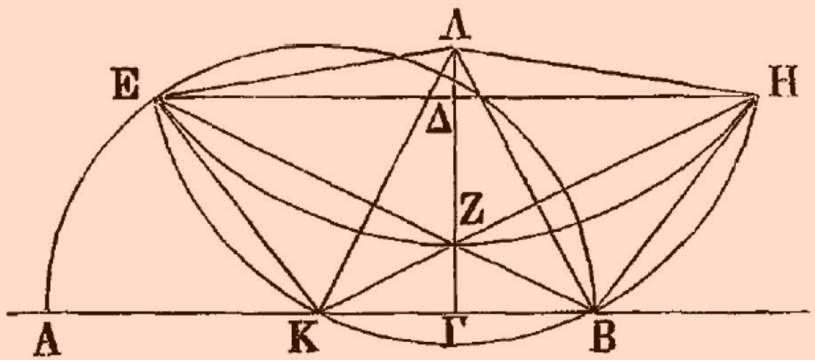


E Next in order he assumes [an outer circumference] greater than a semicircle [obtained by] constructing a trapezium having three sides equal to one another while one, the greater of the parallel sides, is such that the square on it is three times the square on each of those sides, and then comprehending the trapezium in a circle [Here, Simplicios inserts a proof that this can be done] and circumscribing about its greatest side a segment similar to those cut off from the circle by the three equal sides.



That the said segment is greater than a semicircle is clear if a diagonal is drawn in the trapezium. For this diagonal, subtending two sides of the trapezium, must be such that the square on it is greater than double the square on one of the remaining sides. Therefore the square on BΓ is greater than double the square on either BA, AΓ, and therefore also on ΓΔ [A proof is inserted, probably by Simplicios. The diagram letters in the preceding are also likely to have been provided by Simplicios]. Therefore the square on BΔ, the greatest of the sides of the trapezium, must be less than the sum of the squares on the diagonal and that one of the other sides which is subtended by the said [greatest] side together with the diagonal. For the squares on BΓ, ΓΔ are greater than three times, and the square on BΔ is equal to three times, the square on ΓΔ. Therefore the angle standing on the greatest side of the trapezium is acute. Therefore the segment in which it is is greater than a semicircle. And this segment is the outer circumference of the lune [Simplicios here observes that Eudemos omits the actual proof of the squaring of the lune].

F If [the outer circumference] were less than a semicircle, Hippocrates solved



this also, using the following preliminary construction. Let there be a circle with diameter AB and centre K. Let ΓΔ bisect BK at right angles; and let the straight line EX be placed between this and the circumference verging towards B so that the square on it is one-and-a-half times the square on one of the radii. Let EH be drawn parallel to AB, and from K let [straight lines] be drawn joining E and Z. Let the straight line [KZ] joined to Z and produced meet EH at H, and again let [straight lines] be drawn from B joining Z and H. It is then manifest that EZ produced will pass through B – for by hypothesis EZ verges towards B – and BH will be equal to EK.

This being so, I say that the trapezium EKBH can be comprehended in a circle.

G Next let a segment of a circle be circumscribed about the triangle EZH; then clearly each of the segments on EZ, ZH will be similar to the segments on EK, KB, BH.

This being so, the lune so formed, whose outer circumference is EKBH, will be equal to the rectilineal figure composed of the three triangles BZH, BZK, EKZ. For the segments cut off from the rectilineal figure, inside the lune, by the straight lines EZ, ZH are (together) equal to the segments outside the rectilineal figure cut off by EK, KB, BH. For each of the inner segments is one-and-a-half times each of the outer, because, by hypothesis, the square on EZ is one-and-a-half times the square on the radius, that is, the square on EK or KB or BH. Inasmuch then as the lune is made up of the three segments and the rectilineal figure *less* the two segments – the rectilineal figure including the two segments but not the three – while the sum of the two segments is equal to the sum of the three, it follows that the lune is equal to the rectilineal figure.

H That this lune has its outer circumference less than a semicircle, he proves by means of the angle EKH in the outer segment being obtuse. And that the angle EKH is obtuse, he proves thus.

Since $EZ^2 = {}^3/_2 EK^2$

and $KB^2 > 2BZ^2$,

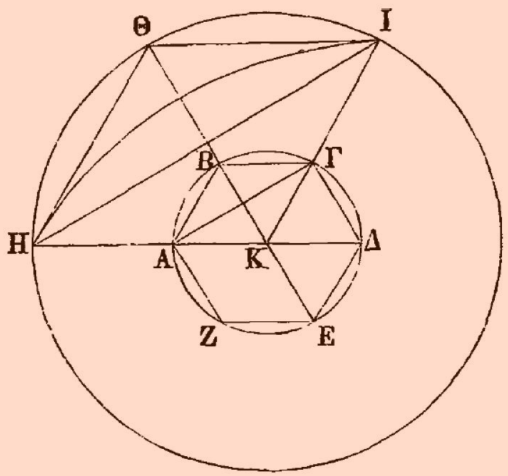
it is manifest that $EK^2 > 2KZ^2$.

Therefore $EZ^2 > EK^2 + KZ^2$.

The angle at K is therefore obtuse, so that the segment in which it is is less than a semicircle.

I Thus Hippocrates squared every lune, seeing that [he squared] not only the lune which has for its outer circumference a semicircle, but also the lune in which the outer circumference is greater, and that in which it is less, than a semicircle [Reviel Netz suspects section (I) to be due to Simplicios; I tend to agree].

J But he also squared a lune and a circle together in the following manner.



Let there be two circles with K as centre, such that the square on the diameter of the outer is six times the square on the diameter of the inner. Let a [regular] hexagon ΑΒΓΔΕΖ be inscribed in the inner circle, and let KA, KB, KΓ be joined from the centre and produced as far as the circumference of the outer circle, and let KA, KB , KΓ be joined. Then it is clear that HΘ, ΘI are sides of a [regular] hexagon inscribed in the outer circle. About HI let a segment be circumscribed similar to the segment cut off by HΘ. Since then $HI^2 = 3\Theta H^2$ (for the square on the line subtended by two sides of the hexagon, together with the square on one other side, is equal, since they form a right angle in the semicircle, to the square on the diameter, and the square on the diameter is four times the side of the hexagon, the diameter being twice the side in length and so four times as great in square), and $\Theta H^2 = 6AB^2$, it is manifest that the segment circumscribed about HI is equal to the segments cut off from the outer circle by HΘ, ΘI, together with the segments cut off from the inner circle by all the sides of the hexagon. For $HI^2 = 3H\Theta^2$, and $\Theta I^2 = H\Theta^2$, while ΘI^2 and $H\Theta^2$ are each equal to the sum of the squares on the six sides of the inner hexagonal, since, by hypothesis, the diameter of the outer circle is six times that of the inner. Therefore the lune HΘI is smaller than the triangle HΘI by the segments taken away from the inner circle by the sides of the hexagon. For the segment on HI is equal to the sum of the segments on HΘ, ΘI and those taken away by the hexagon. Therefore the segments [on] HΘ, ΘI are less than the segment about HI by the segments taken away by the hexagon. If to both sides there is added the part of the triangle which is above the segment about HI, out of this and the segment about HI will be formed the triangle, while out of the latter and the segments

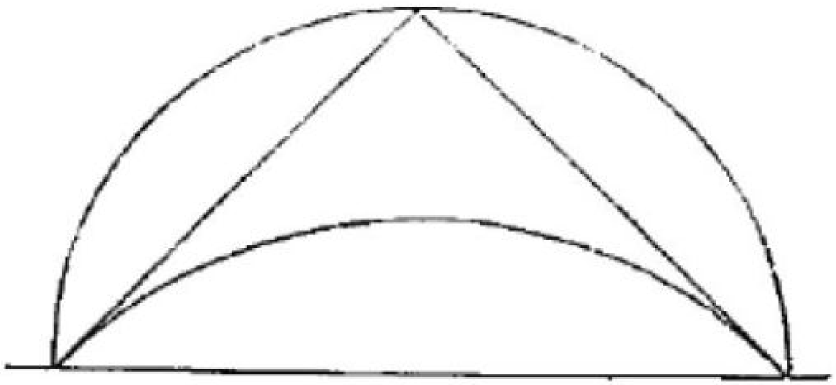
[on] $H\Theta$, ΘI will be formed the lune. Therefore the lune will be less than the triangle by the segments taken away by the hexagon. For the lune and the segments taken away by the hexagon are equal to the triangle. When the hexagon is added to both sides, this triangle and the hexagon will be equal to the aforesaid lune and to the inner circle. If then the aforementioned rectilineal figures can be squared, so also can the circle with the lune.

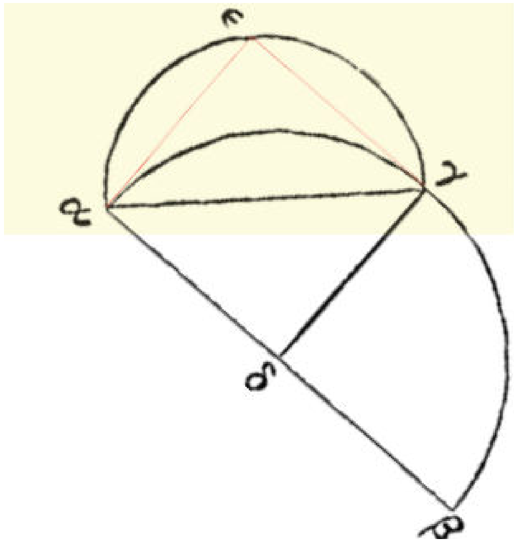
Instead, I shall merely set out what comes out of analysis of the text.

Section C confirms that the proportionality of circular areas to the squares on the diameters is taken for granted.

The argument in section D is not clear, unless once again we make appeal to the well-known geometry of the square (that is, completing the diagram).

> It looks as if section D is a re-elaboration of what is reported from Alexander in section A – a re-elaboration that has deleted part of the traces of the underlying thinking.

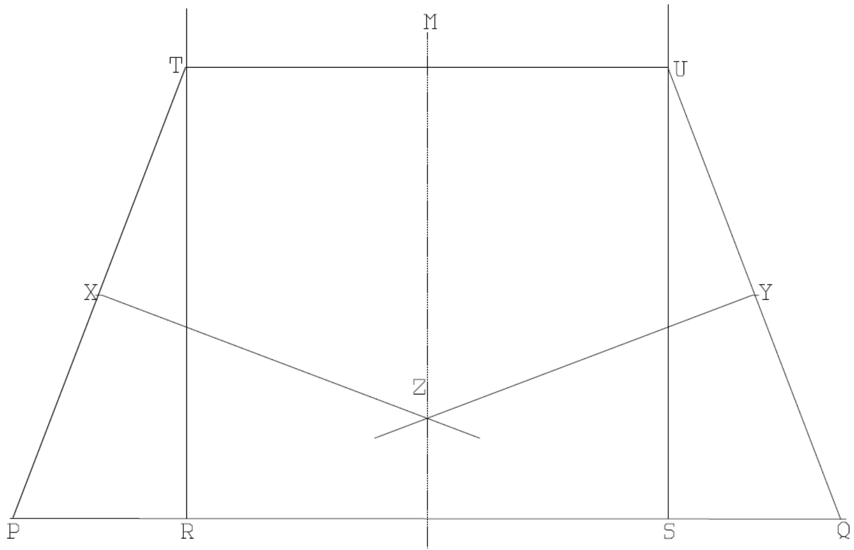




The basis for section D is, once again, the Pythagorean rule and the proportionality of similar areas to the squares on a characteristic linear dimension.

Section E is not quite as simple.

At first, Hippocrates constructs a trapezium with sides s, s, s and $\sqrt{3s^2}$.



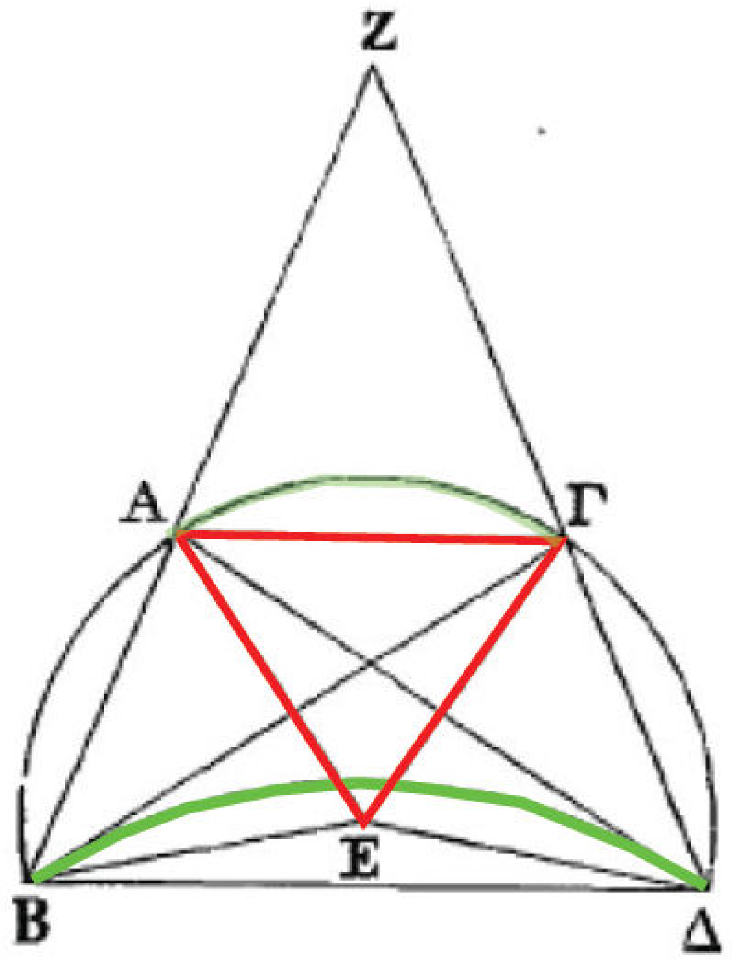
Hippocrates does not explain how to do it, but the necessary tools were:

firstly, the Pythagorean rule, which allows the construction of $\sqrt{3s^2}$;

secondly, the *construction* of a bisecting perpendicular – according to Proclos a recent discovery due to Oinopides.

Construction of the bisecting perpendiculars on PT and QU and symmetry arguments (or symmetry intuition) shows that they meet at Z, which is therefore the centre of a circumscribed circle. Hippocrates again seems just to "know" or "see", he does not explain.

Next a lune has to be constructed. Its outer arc is part of the circumscribed circle, the inner should span the base and be similar to the arcs of this over each of the shorter sides.



Once more, Hippocrates does not tell how to do it, but unless we insist (as does Simplicios) on using the Euclidean definition of similarity, it is easy (it asks for a triangle similar to ΑΓΕ (that is, equilateral) but with base $\sqrt{3} \cdot s$.

The proof that the lune in question spans more than the semi-circle tells us more. It makes use – not, as has been claimed, of the extended Pythagorean theorem but of the principle that if the one side in a triangle is larger than the sum of the others in square, then the opposing angle is larger than right.

Whoever knows the Pythagorean rule will easily understand this; once again, locally obvious knowledge is made use of.

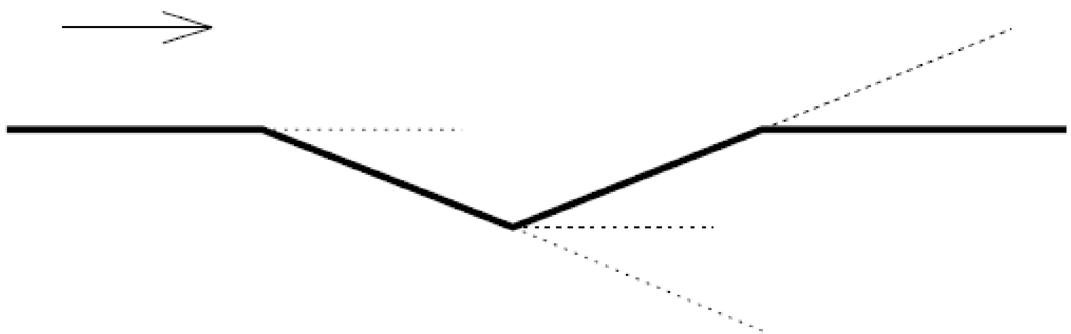
We should add that Greek practical geometry had worked with angle geometry at least since a century when Hippocrates wrote.

That is shown by the famous Samos tunnel, built under the direction of Eupalinos between *ca* 550 and *ca* 530 BCE.

Having to avoid soft rock, Eupalinos at a certain point has to turn some 20° to the right;

– after excavating for a while in this new direction, he turns twice as much to the left, and goes on just as long in this new direction
– and then turns 20° to the right

(thus the unmistakable ideal planning; further complications of the rock has led to some deviations):



Section F constructs a lune on an arc smaller than a semi-circle. In this way Hippocrates has explored how to make lunes belonging to three classes.

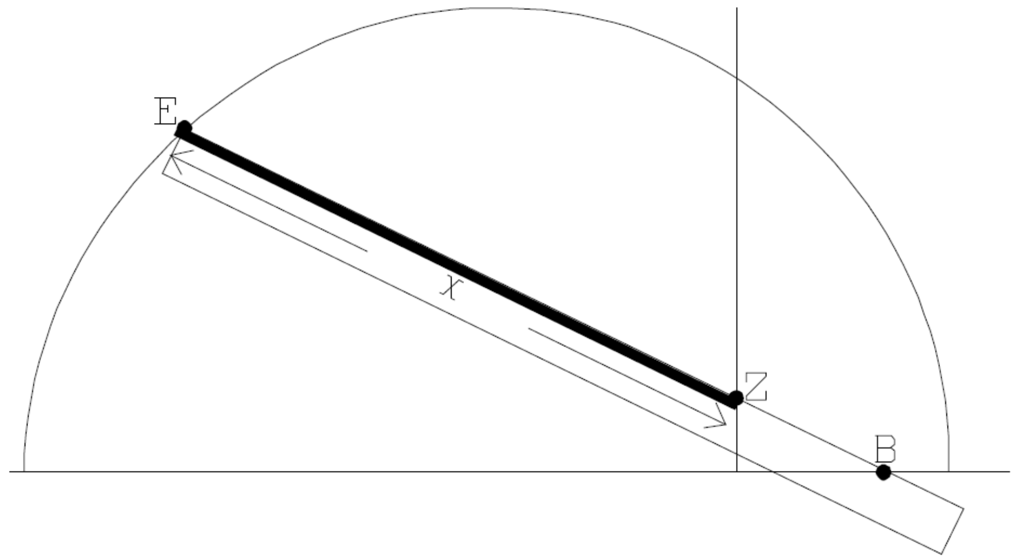
Simplicios's text claims that this means that "Hippocrates squared every lune".

It is undecidable whether Hippocrates thought so.

Alternatively, we may think of those modern mathematicians who proved special cases of Fermat's Last Theorem hoping (in vain) to find thereby a way to prove it in general.

According to Aristotle, some ancient geometers at least seem to have been overly optimistic in one or the other way.

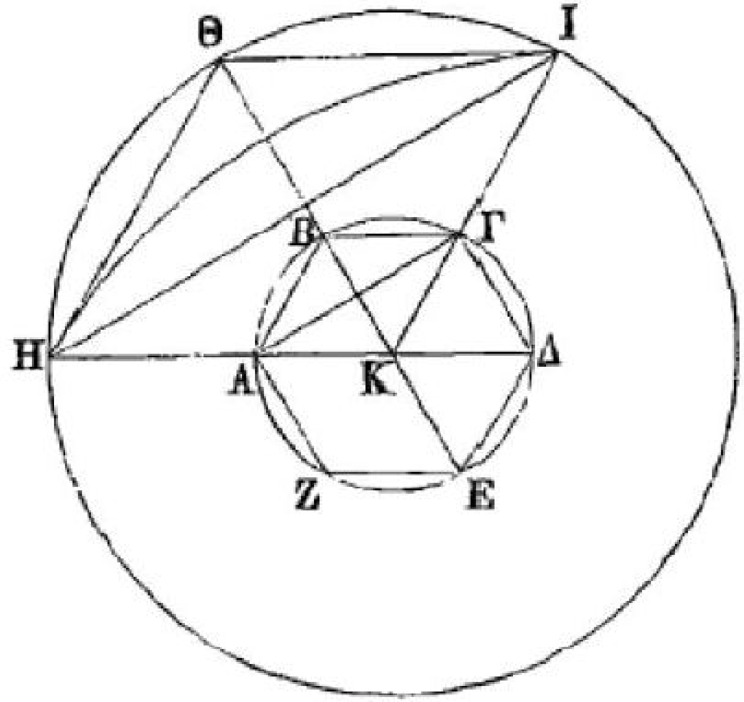
The procedure of section F is quite complex. New compared to the preceding sections is the use of a verging construction.



An unargued claim that certain segments are “clearly” similar may build on the principle that similar segments correspond to similar inscribed angles, apparently also used before.

The “clear” similarity is also left unexplained by Simplicios, who may not have understood (elsewhere he explains simpler matters in detail). After all, he was a philosopher and no geometer.

Section J, the squaring of a lune and a circle together, could be a reparation of Alexander's section B.



It makes use of the same principles:

- the Pythagorean rule;
- the basic geometry of the regular hexagon;
- the proportionality of similar areas to the squares on a characteristic linear extension;
- and simple arithmetic of areas.

It thus teaches us nothing new.

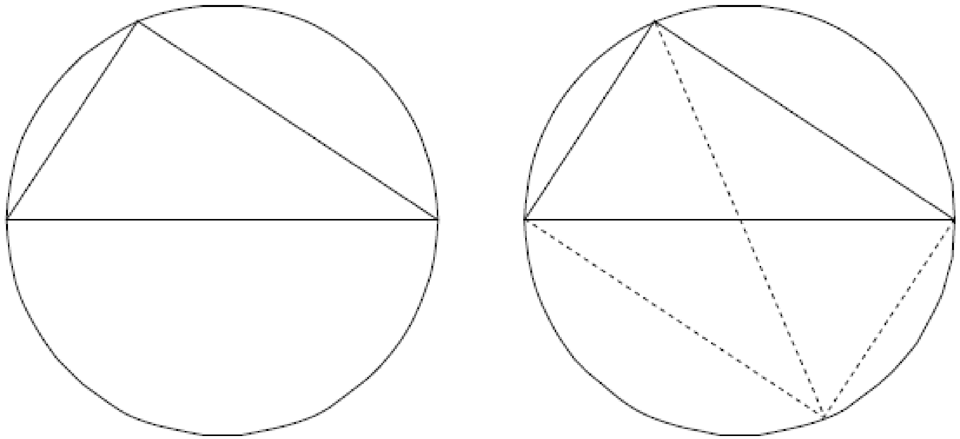
To sum up, let us look at the “tool box” used by Hippocrates.

In the Alexander version, it comprises

- the Pythagorean rule;
- the basic geometry of the regular square and hexagon;
- the proportionality of similar areas to the squares on a characteristic linear extension;
- and simple arithmetic of areas.

To this comes in the Eudemos version some angle geometry, and probably symmetry intuitions.

Part of this – that the diameter of a circle seen from a point on the periphery spans a right angle – was already known to Old Babylonian calculators.



Since Heron knew the characteristic way to express the perimeter of a circle as a repetition of the diameter and not as a multiplication, we know that Old Babylonian Near Eastern geometrical practice was transmitted until classical times.

What else is used concerning angles does not go beyond what is reflected in Eupalinos's planning.

Also recent is the erection of a bisecting perpendicular, which seems to inhere in the unexplained constructions of trapezia with circumcircles.

The erection of a perpendicular could be made by means of a set square, and was probably made in that way (“gnomonwise”) until Oinopides made his construction.

The *bisecting* perpendicular cannot be constructed by means a set square unless we use a foldable string to find the mid-point or have recourse to further tricks;

but it is part of Oinopides’s construction of the perpendicular.

The verging construction, we may observe, represents a string running through an eye and marked by a knot.

In practical geometry, line, circle and verging are all string constructions.

These are the recurrent building blocks, occasionally "pointed at" explicitly. They are the "letters" from which Hippocrates's proofs were composed.

> Whether *techniques* like the drawing of a circle or the verging construction or (at a higher level) the "reduction method" (by which one problem is transformed into another one) were counted by Hippocrates in the same way is undecidable but dubious.

Whether already Hippocrates himself spoke of his building blocks or tool box as στοιχεῖα we cannot know, but they seem to be what would reasonably be collected in the "book" which later times saw as containing "elements".

There is no need, and no reason, to believe that they were organized in anything like a deductive *structure*.

For other purposes, Hippocrates may have used supplementary building blocks; if he could suggest two mean proportionals, he must have known about one mean proportional, and about the proportion concept.

(Not strange, his near-contemporary Archytas knew them as traditional).

These building blocks seem to be referred to or tacitly presupposed in the texts we have as “locally obvious”. Nothing suggests that Hippocrates attempted to produce more than locally coherent deduction.

The end of section B shows what happens when he (or a naive student) tries to iterate deduction:

> There, the outcome of section A, that a particular type of lune can be squared, it taken over as a statement that lunes in general can.

So, asking the question, “how far was Hippocrates on the road toward axiomatization?”, we may answer:

n o t v e r y f a r

Moreover: Nothing suggest that Hippocrates had had the opportunity to become aware that there *was* such a road or such a goal.

Much work was left to the following generations.

10 minutes’ break

Aristotle’s writings are later by some two to three generations. By then, the idea of axiomatizion was at least in the air.

The ideal organization of a field of knowledge as prescribed in the *Posterior Analytic* is obviously inspired by geometry

> “Inspired”, not copying, already for the reason that Aristotelian syllogistic logic does not fit the way geometric proofs are argued.

– but inspired not just *by reasoned* but *by axiomatic* geometry.

During the small century or so that had passed since Hippocrates wrote his elements, many things had changed, and Aristotle presents much material elucidating the process.

Quite a few of Euclid’s definitions (or alternatives referred to by commentators) were known to Aristotle.

Two examples:

Firstly, *Topica* refers to those who define the line as a “length without breadth”, μῆκος ἀπλατὲς, exactly Euclid’s definition I.1.

Secondly, though paraphrased and contracted, the definition of geometrical similarity referred to in *Analytica posteriora* is obviously the one offered in *Elements* VI.

Definitions had been a concern in Greek philosophy for quite some time.

According to Aristotle's *Metaphysics*, "Socrates [...] fixed thought for the first time on definitions".

Whether he was really the first or inspired by contemporary mathematicians is probably not to be decided – not least because Aristotle speaks of ὁρισμοί but Euclid of ὅροι, rather meaning "delimitations".

Aristotle is likely to have been aware that the difference was more than just a choice between synonyms.

Among Euclid’s common notions, the third (“if equals be subtracted from equals, the remainders are equal” is Aristotle’s paradigm for an axiom or “peculiar truth” valid within a particular genus.

> It serves twice as such in *Analytica posteriora*, and also in *Analytica priora* as an example of a presupposition that has to be made explicit in order to avoid a *petitio principii*.

Further, Aristotle knew Euclid’s second postulate – that can be seen in the *Physica*,

> [mathematicians] do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.

Euclid similarly requests (that is the meaning of “postulate”) that it be possible “to produce a finite straight line continuously in a straight line”.

As far as I know, the other postulates are not quoted (nor paraphrased) in the Aristotelian corpus.

One, moreover, is absent where it would have been adequate to mention it, namely in a passage in the *Analytica priora*. The passage refers, as an example of hidden circular reasoning, to

> those persons [...] who suppose that they are constructing parallel straight lines: for they fail to see that they are assuming facts which it is impossible to demonstrate unless the parallels exist.

Aristotle also tells the way out: that one has to *request* ("postulate") that which one cannot prove.

Postulate 5

> if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles

was obviously meant to repair that calamity, in agreement with Aristotle's advice.

Actually, it only does so halfway. It excludes hyperbolic but not elliptic geometries (precisely those where parallels do not exist).

For this purpose, one has to presuppose, for example, that two straight lines cannot enclose a space, which some geometers indeed added as an axiom,

and which in fact is used in a dubious passage in the proof of *Elements* I.4 (almost certainly a scholion that has crept into the text).

What follows from these observations? In general that geometry as known to Aristotle was already striving for axiomatization.

No wonder, we know from Eudemos as quoted by Proclos that at least Theudios made a new, “better arranged” collection of elements.

> This might well mean that it was arranged in a tentatively axiomatic way or at least in longer deductive chains.

We further know from Eudemos through Proclos that a number of mathematicians worked together at the Academy in Plato’s time.

But we also see that the enterprise had not yet led to the goal, at least not as a social undertaking

- those who undertook to construct parallel straight lines while presupposing unconsciously that such lines exist were still building their reasoning on the locally obvious.

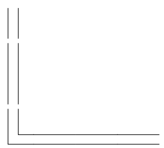
So does even Euclid in many cases, for instance taking it for granted that two lines cannot enclose a space – in other words, that only one straight line connects two points (not to speak of his many topological intuitions).

We may also have a look at Euclid's postulate 4

"That all right angles are equal to each other".

For us, this is locally obvious – "of course, they are all 90°".

Apparently, it was just as obvious until the mid-fifth century BCE – and for a similar reason, they were all made by the set square, the *gnomon:*



Then, according to Proclos, Oinopides introduced the *construction* of a pendicular by means of ruler and compass,

calling the perpendicular a line drawn "gnomonwise" – showing that until then it had indeed been made by means of the set square.

However, with the new construction arose the need for a *definition* of what a right angle is.

In Euclid we find this:

> When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

This seems to solve the problem, now we know what a right angle is, much better than the Old Babylonian surveyor-scribes (and probably the surveyor-scribes of the mid-first millennium), and their distinction between “wrong” and “right” angles.

But it creates a new problem: Now it is no longer obvious that all right angles are equal, and that is needed in many proofs.

The preceding three paragraphs lapsed into old-style historiography of mathematics,

that is, the type of historiography that tended to forget that mathematical knowledge and practice do not exist per se but have social carriers

– or, if mentioning persons, would take it for granted that these, as "mathematicians", would think "like mathematicians".

(That is indeed how the received image of Hippocrates was constructed.)

If you had no objections you will recognize how easily this lapse occurs.

Yet a problem is only one if it is a problem *for somebody*,

and it only becomes a problem in the encounter with that somebody.

Here, we may look at Plato's *Erastae*.

It tells about two boys leaving their school (Plato's own school when he was a boy).

They are in an eager discussion about the ecliptic, apparently its declination.

It they could discuss eagerly about Oinopides and his work on the obliquity of the ecliptic, they might also challenge their teacher,

and ask (this was shortly after Oinopides introduced his construction) *what* this right angle is *in itself* which he constructs (apart from being supposedly useful in astronomy, as Proclos says Oinopides had thought).

The answer might be something like the Euclidean definition.

And then, at a later moment, similar eager students might discover that with this definition, the equality of right angles is no longer obvious.

This is *critique*, born as an endeavour from the character of the environment.

We may further remember that the environment of philosophers (to which we may count the theory-oriented mathematicians teaching elite youth just as did other philosophers) did not strive for truth in peaceful collaboration

but in competition and strife. Here, *critique* would coincide with *criticism* or *challenge* of colleague-competitors.

There is thus no need for critical students,
colleagues would do just as well, or better.

Axiomatization

Critique had been a driving force in the axiomatization of geometry – axiomatization *as a goal* had not been imaginable when Oinopides and Hippocrates made their work.

Not only was axiomatization the outcome of a process yet in their future; so was the discovery of the *idea* of axiomatization as a possibility.

Plato's reproach to geometricians in the *Republic*, that they are

> dreaming about being, but the clear waking vision of it is impossible for them as long as they leave the assumptions which they employ undisturbed and cannot give any account of them.
>
> For where the starting-point is something that the reasoner does not know, and the conclusion and all that intervenes is a tissue of things not really known, what possibility is there that assent in such cases can ever be converted into true knowledge or science?

– this reproach may look as if Plato had observed the strivings of contemporary mathematicians to achieve axiomatic order.

In principle the "assumptions"/ὑπόθεσις he speaks about may also be local, as in Hippocrates's text: but the reference to "all that intervenes" suggests at least deductive chains.

Anyhow, whether an axiomatic structure or just locally coherent argument is meant, Plato does not accept such geometry as more than a mere mental exercise preparing the best souls for the study of dialectics,

> the only process of inquiry that advances in this manner, doing away with hypotheses, up to the first principle itself in order to find confirmation there,

which first principle is insight in "the good", no formulated axiom – apparently mystical insight; and dialectic as imagined by Plato is in consequence no axiomatic system.

Aristotle understood that this was a pipe dream, and that explicit axiomatization is the maximum that can be achieved.

This is indeed pointed out in the very first sentences of his *Analytica posteriora:*

> All instruction given or received by way of argument proceeds from pre-existent knowledge. This becomes evident upon a survey of all the species of such instruction.
>
> The mathematical sciences and all other speculative disciplines are acquired in this way, and so are the two forms of dialectical reasoning, syllogistic and inductive;
>
> for each of these latter makes use of old knowledge to impart new, the syllogism assuming an audience that accepts its premisses, induction exhibiting the universal as implicit in the clearly known particular.

In spite of the ambiguity of Plato's polemics, these words together with the rest of the *Analytica posteriora* leave no doubt that in the mid-fourth century BCE not only Aristotle but also the geometers were familiar with the axiomatic *ideal*.

From now on, it provided a possible format when new fields were taken up and did not need to be the unplanned outcome of a process driven by other forces.

As we know, this format was to be used for example by Archimedes, however much his *aim* was *new knowledge* or astonishing *results*, not system-building.

For mathematicians at the level of Archimedes and Apollonios, reasoning had to build on the already existing axiomatic system – *their* locally obvious included Euclid's *Elements*
(or in the case of Archimedes, something very much alike).

Critique, as I argued, had been a motive force in the process ending up in axiomatization before this process could be driven by a recognized aim.

But critique was more than that.

A look at *Elements* II.6 will illustrate this:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

Whoever encounters these lines for the first time is likely to ask why this seemingly abstruse theorem is interesting. It is used by Apollonios, but that could not be foreseen when the *Elements* were put together.

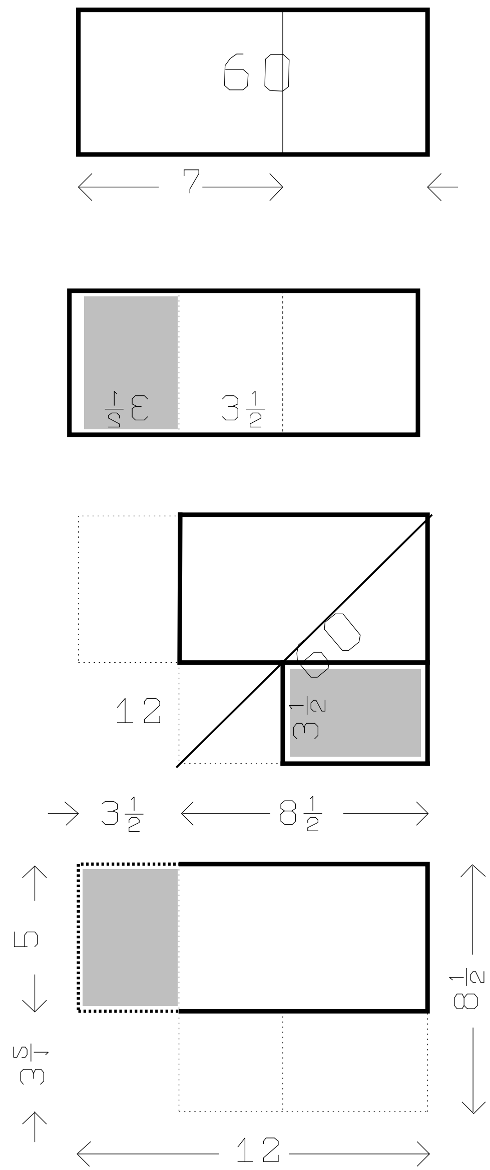
We may look instead of how surveyors had solved the riddle of finding the sides of a rectangle from its area and the difference between length and width. I shall recycle the *igûm-igibûm* problem.

The diagram shows a rectangle, whose area is 60, and $L–W = 7$.

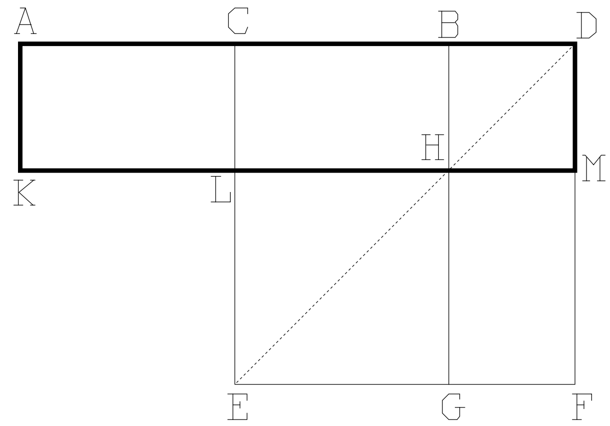
That part of the rectangle that exceeds the square □(W) is bisected, and the outer part (with width 3½) moved around so as to form together with what remains in place a gnomon – still with area 60.

The square lacking in the lower left corner has the area 3½ × 3½ = 12¼. The completed square therefore has the area 60+12¼ = 72¼, and hence side 8½.

Putting the part which was moved back into its original position we find that the length will be 8½+3½ = 12, and the width 8½–3½ = 5.



Euclid's diagram is very similar – the only difference being the diagonal which is added to the traditional diagram.

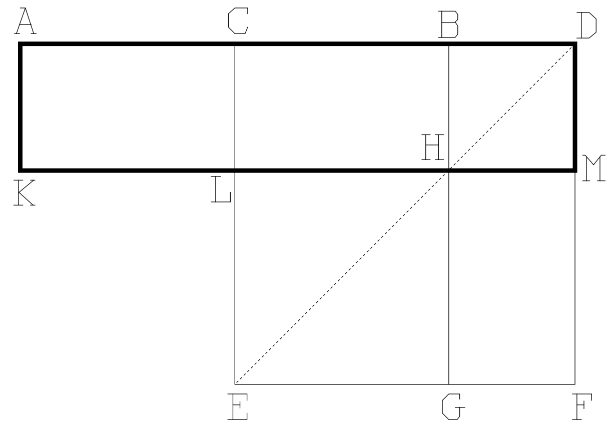


The bisected line is represented by *AB* and the added line by *BD*. *DM*, perpendicular to *AD*, equals *BD*. *AB* is thus the excess of length over width in the rectangle *ADMK*. If we identify *AD* with *L* and *DM* with the *W*, we are back at the traditional problem, and *AB* must be 7.

There are differences, however. Firstly, Euclid does not solve a problem: if we impress algebraic categories on his text, then he presents us with an identity.

This identity can of course be used to solve problems by taking some of the magnitudes involved to be known and others unknown (for example, taking *AB* to be 7 and the area *ADMK* to be 60 will allow us to find *AD* and *BD*).

Secondly, Euclid does not move segments or areas around. At first he *constructs* the square *CDFE* on *CD*, which ensures that the angle *ADF* is really a right angle. He then draws the diagonal *DE*, which has no place in the traditional procedure.



He then draws the line *BG* parallel to *CE* or *DF;* through the intersection *H* of *BG* and *DE* he draws *KM* parallel to *AB* or *EF*, and through *A* the line *AK* parallel to *CL* or *DM*. That allows Euclid to show that rectangle *ACLK* is equal to the rectangle *HMFG*, and thus that the rectangle *ADMK* equals the gnomon *CDFGHL*, whence finally the equality claimed in the enunciation.

Nothing is cut, moved around and pasted, all is proved to the best standards of theoretical geometry as these had been shaped in the late fifth and the fourth centuries BCE.

The proposition thus functions as a critique of the cut-and-paste procedure by which the problem was traditionally solved,

showing *why* and *under which precisely stated conditions* it works – thus saving it instead of rejecting it as Plato did when he reproached geometers their “talk of squaring and applying and adding and the like”.

That it was also *meant* as critique and saving appears to follow from analysis of the whole sequence *Elements* II.1–10. A discussion in depth would lead too far. A blunt summary goes like this:

- All 10 propositions correspond in the way just sketched for II.6 to riddles or basic cut-and-paste-tricks belonging at least since ca 1800 BCE to an environment of surveyors – riddles which once inspired the Old Babylonian scribe school, but have also left their traces in a variety of written mathematical cultures until the Late Middle Ages, including Greek pseudo-Heronic practical geometry (and were therefore certainly known to Greek theoretical geometers);

- propositions 4–7 are used later in the *Elements*, mainly in Book X, the others not; like many of the definitions of Book I that are never used afterwards, they represent something familiar that has to be saved for its own sake;

- propositions 2 and 3 are special cases of proposition 1; propositions 4 and 7 are different formulations of what is practically the same matter; the same can be said about propositions 5 and 6 and about propositions 9 and 10.

  None the less, all are proved independently, as if not only the results but also the traditional methods had to be saved through critique.

So, between Aristotle's and Euclid's times, *deductivity completed as axiomatization* established itself as the norm for how mathematics should be made

– obviously only within the tiny group which we, like Netz, would normally accept as "mathematicians".

Most of those who went through the normal syllabus of Liberal Arts would not care about anything beyond rhetoric, as already said

Within that minority which had greater ambitions, most would stop at knowing a few concepts and enunciations and not care for demonstrations.

> That is clear from the relative popularity of Nicomachos's writings, from handbooks like those of Martianus Capella and Cassiodorus, and from Theon of Smyrna's explanation of the mathematics needed for the study of Plato.

Among those who calculated or constructed for administrative or productive purposes, the norm never took root, at most we find arguments from the locally obvious – to see this, we may look at Vitruvius and the pseudo-Heronic writings.

In Euclid's time already, the effects of the "liberal" curiosity of the fifth century BCE had subsided and been replaced by institutionalized norms.

For that reason, the importance of critique as a partner and root of axiomatization seems also to have subsided

> (after all, the critique in *Elements* II is almost certainly borrowed from late fifth or early fourth-century predecessors, as the proportion theory in *Elements* V is supposed to be borrowed from Eudoxos).

Heron's *Metrica* may to some extent be considered a rewriting of practical geometry "from a higher vantage point" – but only to a quite limited extent in a way that allows us to speak of critique.

And then?

Not too long after Euclid's third century BCE, Greek mathematics entered the age of commentaries or, using Reviel Netz's term, of "deuteronomic texts"

> (a somewhat broader category, encompassing also summaries and such things).

In Simplicios's presentation of the Hippocratic fragment (early sixth century CE) he states:

> I shall set out what Eudemos wrote word for word, adding only for the sake of clearness a few things taken from Euclid's *Elements* on account of the summary style of Eudemos, who set out his proofs in abridged form in conformity with the ancient practice.

That illustrates a partial change of norms.

Commentaries fill out and explain; at times they also discuss.

Even though Simplicios is engaged in a commentary to Aristotle, he follows the commentator habits and norms even here,

but mainly by filling out and, implicitly, explaining.

“Adding [...] a few things from Euclid’s *Elements*” means that Simplicios inserts the Hippocratic text in the axiomatized framework.

In its own way, the addition of commentaries and the standardized structuring of mathematical texts is a new level of critique,

arguing now *why and in which sense* the classical text that is commented upon is right and conformable to norms.

But since this classical text has somewhat sacred status, this critique is uncritical

– quite different from the critical critique of fellow-philosophers or teachers in the fifth to fourth centuries BCE.

Genuinely critical stances had not disappeared – but they had become external, attacking the whole undertaking, not trying to save or to find the “possibility and limits” of mathematical knowledge.

The best example is probably Sextus Empiricus. This is harsh but informative and informed criticism – but not critique.

Norms certainly govern practice only to some extent;

many causes – conflicting norms, incompetence, personal conflicting interest, and so forth – make actors deviate from them.

Eudemos's lack of reference to the propositions which Simplicios feels he needs to insert may be due to fidelity to his source – he is writing a history of geometry and may have written more like a historian than as a mathematician.

But it may also reflect that axiomatization in his time was still a developing *practice* and not yet fully effective as a norm.

In Simplicios's age of deuteronomic texts, in contrast, the norm had become so explicit that we may see it as an *ideology*, an inextricable amalgam of the descriptive and the prescriptive, of "is" and "ought to".

From "is" to "ought to"

"Because I'm your mother. That's why!"

That ideology is still with us, admittedly more effective when governing the writing of textbooks (deuteronomic, indeed) than in mathematical research.

This ideology not only fuses the descriptive and the prescriptive levels.

It also corresponds to the interpretation of ideology as “false consciousness”.

Most obviously, it disregards informatics, quantitatively the major part of 21st-century mathematics. Already in 1970, a textbook from that field declared that

> It is a commonplace that numerical processes that are efficient usually cannot be proven to converge, while those amenable to proof are inefficient [...]. The best demonstration of convergence is convergence itself.

This was written at a moment when the students using the book were supposed to work in FORTRAN, PL/1 or ALGOL – when programming was thus still transparent compared to what we find today.

Every time your computer screen freezes, remember that the reason is almost certainly an unanticipated conflict somewhere on the path from machine code through compiler to operating system and application

thus proof that the software has not been derived axiomatically from first principles.

The role of beta-versions is to locate the conflicts (“bugs”) that are most likely to occur – doing so empirically, however;

but this “critique through practice” never succeeds in doing more. The demonstrations of algorithm design remain local.

Even if we try to save the honour of mathematics by excluding informatics, the ideology misrepresents reality.

In 1545, Cardano's *Ars magna* was printed. Then, gradually, the power of the tools offered by Descartes' *Géométrie* (1637) (also in analysis of the infinite and the infinitely small) was revealed.

First, this transformed fundamentally what *algebra* could be; soon it also changed the global character of mathematics.

Until the late 19th century, this whole process was founded (when not on controlled guess, as often happened) on arguments and demonstrations of no more than "local" validity,

> that is, premises that it seemed reasonable to accept or at least to try, but which were not built on clearly formulated first principles.

Critique gradually improved the situation (even this was an epoch of competing scholars),

but only the late 19th century was once again able to reshape mathematics on an axiomatic footing.

In its merger of description and prescription, the ideology of thorough demonstration and demonstrability thus becomes false consciousness.

The prescriptive aspect not only imposes a particular interpretation of the facts on the description – that probably cannot be avoided. It distorts it in a way that is easily looked through *if only one wants to*.

> Some years ago in Italy, a nun when told by physicians that her supposed stomach ache were birth pangs, exclaimed "it is not possible, I am a nun!" Her false consciousness cannot have survived the next few hours.

In general, false consciousness survives on Darwinian conditions: in some way it has to be useful.

The one we have looked at here provides mathematics (that is, the mathematical establishment) with a comforting self-image which can be projected.
(while the inconvenient baby, informatics, is given into adoption).

It also serves to ostracize mathematical cultures that deviate from what the ideology prescribes and what is therefore claimed to *de*scribes “Western mathematics”;

thereby it serves a more direct and more indisputably political “projection of power”.