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Old Babylonian supra-utilitarian mathematics – in particular the so-called "algebra"

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Some background

Last time I presented a socio-cultural history of Mesopotamian mathematics from the fourth-millennium beginnings until the end of the Assyrian Empire (*ca* 600 BCE), without going into the mathematics itself.

Today I shall speak instead of the supra-utilitarian “algebra” of the second half of the Old Babylonian epoch, 1800–1600 BCE.

First, however, some tools.

Writing

At least from Early Dynastic times, writing had been in Sumerian (before that the purely logographic writing system without grammatical or phonetic complements does not allow us to decide).

Signs retained their logographic or ideographic value, but were also used according to their phonetic value, not least as grammatical complements (the “rebus principle”.

For instance, the verb “say” was pronounced more or less as the possessive suffix “its”, for which reason the same sign was used.

Through reduction, similar signs with different logographic meanings might also end up being written in the same way. Thereby the same sign might have several logographic and several phonetic meanings.

When the writing language changed from Sumerian to Akkadian in Old Babylonian times, this became even more complicated.

When cuneiform texts are transliterated, words to be read as logograms or in Sumerian are transliterated in SMALL CAPS.

Phonetic Akkadian writing is transcribed as *italics*.

Sometimes, Assyriologists transcribe uninterpreted signs by their “sign name” (mostly a *possible* Sumerian reading, sometimes a trace of protoliterate understanding) in VERSALS. I shall not do that here.

The sexagesimal place-value system

The Old Babylonian mathematical texts make use of a place-value number system with base 60 with no indication of a “sexagesimal point” – it was “floating-point”.

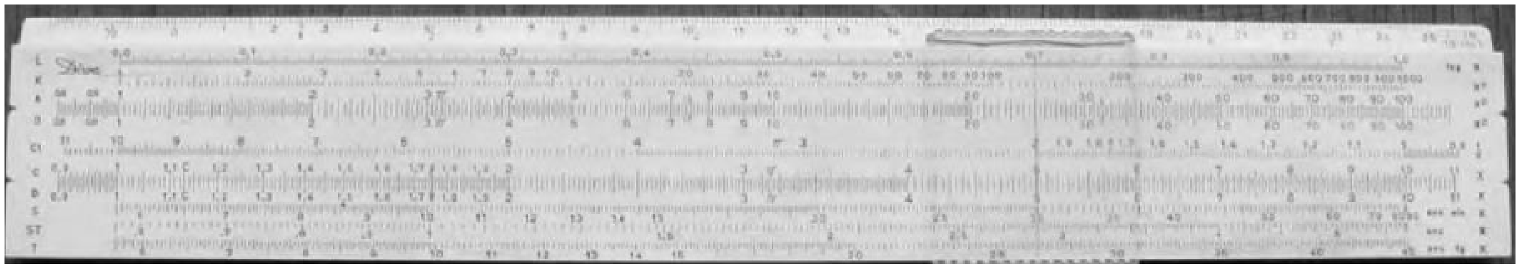
In our notation, which also employs place value, the digit “1” may certainly represent the number 1, but also the numbers 10, 100, ..., as well as 0.1, 0.01, ... Its value is determined by its distance from the decimal point.

Similarly, “45” written by a Babylonian scribe may mean 45; but it may also stand for $\frac{45}{60}$ (thus $\frac{3}{4}$); for $45 \cdot 60$; etc. No decimal point determines its “true” value.

The system corresponds to the slide rule of which engineers made use of before the arrival of the electronic pocket calculator.

This device also had no decimal point, and thus did not indicate the absolute order of magnitude.

In order to know whether a specific construction would ask for 3,5 m^3, 35 m^3 or 350 m^3 of concrete, the engineer had recourse to mental calculation.



For writing numbers between 1 and 59, the Babylonians made use of a vertical wedge (𒁹) repeated until 9 times *in fixed patterns* for the numbers 1 to 9, and of a *Winkelhaken* (a German loanword originally meaning “angular hook”) (𒌋) repeated until 5 times for the numbers 10, 20, ..., 50.

In translations of Babylonian mathematical texts it is therefore customary to indicate the order of magnitude that has to be attributed to numbers. Several methods are in use.

Here I shall employ a generalization of the degree-minute-second notation. If 𒌋𒐊 means $\frac{15}{60}$, I shall transcribe it 15´;
if it corresponds to $\frac{15}{60\cdot60}$, I shall write 15´´.

If it represents 15·60, I write 15`, etc. If it stands for 15, I write 15 or, if needed in order to avoid misunderstandings, 15°.

 understood as $10+5\cdot60^{-1}$ will thus be transcribed $10°5´$

 understood as $30´$ thus means $\frac{1}{2}$.

 understood as $45´$ means $\frac{3}{4}$.

 understood as $12´$ means $\frac{1}{5}$; understood as $12`$ it means 720.

 understood as $10´$ means $\frac{1}{6}$.

 may mean $16`40 = 1000$ or $16°40´ = 16\frac{2}{3}$, etc.

 may mean $1`40 = 100$, $1°40´ = 1\frac{2}{3}$, $1´40´´ = \frac{1}{36}$, etc.

Outside school, the Babylonians employed the place-value system exclusively for intermediate calculations (exactly as an engineer used the slide rule in 1970).

When a result was to be inserted into a contract or an account, they could obviously not allow themselves to be ambiguous; other notations allowed them to express the precise number they intended.

Now to our central topic

Let us look at a very simple example extracted from a text written during the 18th century BCE in transliteration:

1. A.ŠÀ$^{\mathrm{l}[\mathit{am}]}$ *ù mi-it-ḫar-ti ak-m*[*ur-m*]*a* 45-E 1 *wa-ṣi-tam*
2. *ta-ša-ka-an ba-ma-at* 1 *te-ḫe-pe* [3]0 *ù* 30 *tu-uš-ta-kal*
3. 15 *a-na* 45 *tu-ṣa-ab-ma* 1-[E] 1 ÍB.SI$_8$ 30 *ša tu-uš-ta-ki-lu*
4. *lìb-ba* 1 *ta-na-sà-aḫ-ma* 30 *mi-it-ḫar-tum*

Around 1930, when the technical terminology had not yet been cracked, this was as opaque to historians of mathematics as it is to you.

The sequence of numbers, however, provides possible cue:

$$45' \; (= \tfrac{3}{4}) — 1° — 1° — 30' \; (= \tfrac{1}{2}) — 30' — 15' \; (= \tfrac{1}{4}) — 45' —$$
$$1° — 1° — 30' — 1° — 30'.$$

Compare with this:

$$\begin{aligned} x^2+1\cdot x = \tfrac{3}{4} \Leftrightarrow \quad & x^2+1\cdot x+(\tfrac{1}{2})^2 = \tfrac{3}{4}+(\tfrac{1}{2})^2 \\ \Leftrightarrow \quad & x^2+1\cdot x+(\tfrac{1}{2})^2 = \tfrac{3}{4}+\tfrac{1}{4} = 1 \\ \Leftrightarrow \quad & (x+\tfrac{1}{2})^2 = 1 \\ \Leftrightarrow \quad & x+\tfrac{1}{2} = \sqrt{1} = 1 \\ \Leftrightarrow \quad & x = 1-\tfrac{1}{2} = \tfrac{1}{2} \end{aligned}$$

The central trick is the quadratic completion.

The same numbers occur in almost the same order. The same holds for many other texts.

In the early 1930s historians of mathematics thus became convinced that between 1800 and 1600 BCE the Babylonian scribes knew something very similar to our equation algebra.

The next step was to interpret the texts precisely.

To some extent, the general, non-technical meaning of their vocabulary could assist.

In line 1 of the problem , *ak-mur* may be translated “I have heaped”. An understanding of the “heaping” of two numbers as an addition seems natural and agrees with the observation that the “heaping” of 45´ and 15´ (that is, of $\frac{3}{4}$ and $\frac{1}{4}$) produces 1.

When other texts “raise” (*našûm*) one magnitude to another one, it becomes more difficult. However, one may observe that the “raising” of 3 to 4 produces 12, while 5 “raised” to 6 yields 30, and thereby guess that “raising” is a multiplication.

In this way, Otto Neugebauer, François Thureau-Dangin and others came to choose a purely arithmetical interpretation of the operations – that is, as additions, subtractions, multiplications and divisions *of numbers*.

This translation offers an example:

1. I have added the surface and (the side of) my square: 45´.
2. You posit 1°, the unit. You break into two 1°: 30´. You multiply (with each other) [30´] and 30´:
3. 15´. You join 15´ to 45´: 1°. 1° is the square of 1°. 30´, which you have multiplied (by itself),
4. from 1° you subtract: 30´ is the (side of the) square.

Such translations are still found today in general histories of mathematics.

They explain the numbers that occur in the texts, and they give an almost modern impression of the Old Babylonian methods. There is no fundamental difference between the present translation and the solution by means of equations.

If the side of the square is x, then its area is x^2. Therefore, the first line of the text – the problem to be solved – corresponds to the equation $x^2+1\cdot x = \frac{3}{4}$.

Continuing the reading of the translation we see that it follows the symbolic transformations step by step.

But even though the present translation as well as others made according to the same principles explain the numbers of the texts, they agree less well with their words, and occasionally not with the order of operations.

Firstly, these translations do not take the geometrical character of the terminology into account, supposing that words and expressions like “(the side of) my square”, “length”, “width” and “area” of a rectangle denote nothing but unknown numbers and their products.

Further, the number of operations is too large.

For example, there are two operations that in the traditional interpretation are understood as addition: “to join to” (*waṣābum*/DAḪ), and “to heap” (*kamārum*/ GAR.GAR).

Are these simply synonym, as believe by the fathers?

Certainly, we too know about synonyms even within mathematics – for instance, “and”, “added to” and “plus”;

the choice of one word or the other depends on style, on personal habits, on our expectations to the interlocutor, and so forth.

Synonyms can also be found in Old Babylonian mathematics.

Thus, the verbs “to tear out” (*nasāḫum*/ZI) and “to cut off” (*ḫarāṣum*/KUD) are names for the same subtractive operation: they are used in strictly analogous situations.

The difference between “joining” and “heaping”, however, is of a different kind.

No text exists which refers to a quadratic completion as a “heaping”. “Heaping”, on the other hand, is the operation to be used when an area and a linear extension are added.

These are thus distinct operations, not two different names for the same operation.

In the same way, there are two distinct “subtractions”, four “multiplications”, and even two different “halves”.

A translation which mixes up operations which the Babylonians treated as distinct may explain why the Babylonian calculations lead to correct results; but they cannot penetrate Babylonian mathematical thought.

Further, the traditional translations had to skip certain words which seemed to make no sense.

A more literal translation of the last line of our small problem would begin “from the inside of 1°” (or even “from the heart” or “from the bowels”). Not seeing how a number 1 could possess an “inside” or “bowels”, the translators tacitly left out the word.

Other words were translated in a way that differs so strongly from their normal meaning that it must arouse suspicion.

Normally, the word translated “unity” by Thureau-Dangin and “coefficient” by Neugebauer (*waṣītum*, from *waṣûm*, “to go out”) refers to something that sticks out, as that part of a building which architects speak about as a “projection”.

That appeared absurd – how can a number 1 “stick out”? Better make the word correspond to something known in the mathematics of our own days!

Finally, the order in which operations are performed is sometimes different from what seems natural in the arithmetical reading.

A genuine interpretation – a reading of what the Old Babylonian calculators thought and did – must take two things into account:

on one hand, the results obtained in 1930s in “first approximation”;

on the other, the levels of the texts which were neglected by then.

Representation and “unknowns”

In *our* algebra we use x and y as substitutes or names for unknown *numbers*.

We use this algebra as a tool for solving problems that concern other kinds of magnitudes, such as prices, distances, energy densities, etc.;

but in all such cases we consider these other quantities as *represented* by numbers. For us, numbers constitute the *fundamental representation*.

With the Babylonians, the fundamental representation was geometric. Most of their “algebraic” problems concern rectangles with length, width and area, or squares with side and area.

We shall certainly encounter a problem that asks about two unknown *numbers*, but since their product is spoken of as a “surface” it is evident that these numbers are *represented* by the sides of a rectangle.

An important characteristic of Babylonian geometry allows it to serve as an "algebraic" representation: it always deals with *measured* quantities.

The measure of a segment or an area may be treated as *unknown* – but even then it exists as a numerical measure, and the problem consists in finding its value.

Units

Every measuring operation presupposes a metrology, a system of measuring units.; the numbers that result from it are concrete numbers.

That cannot be seen directly in the problem that was just quoted; mostly, the mathematical texts do not show it since they make use of the place-value system (except, occasionally, when given magnitudes or results are stated).

In this system, all quantities of the same kind were measured in a “standard unit” which, with very few exceptions, was not stated but tacitly understood.

The standard unit for *horizontal distance* was the NINDAN, a “rod” of c. 6 m.

In our problem, the side of the square is thus $\frac{1}{2}$ NINDAN, that is, c. 3 m.

For *vertical distances* (heights and depths), the basic unit was the KÙŠ, a “cubit” of $\frac{1}{12}$ NINDAN (that is, c. 50 cm).

The standard unit for *areas* was the SAR, equal to 1 NINDAN^2.

The standard unit for volumes had the same name: the underlying idea was that a base of 1 NINDAN^2 was provided with a standard thickness of 1 KÙŠ.

In agricultural administration, a better suited area unit was used, the BÙR, equal to 30\` SAR, c. 6½ ha.

The standard unit for *hollow measures* (used for products conserved in vases and jars, such as grain and oil) was the SÌLA, slightly less than one litre.

In practical life, larger units were often used: 1 BÁN = 10 SÌLA, 1 PI = 1` SÌLA, and 1 GUR, a “tun” of 5` SÌLA.

Finally, the standard unit for *weights* was the shekel, c. 8 gram. Larger units were the *mina*, equal to 1` shekel (thus close to a British pound) and the GÚ, “a load” equal to 1`` shekel, c. 30 kilogram.

There are two additive operations.

One (*kamārum*/UL.GAR/GAR.GAR), as we have already seen, can be translated "to heap *a* and *b*",

the other (*waṣābum*/DAḪ) "to join *j* to *S*".

"Joining" is a concrete operation which conserves the identity of *S*. If a geometric operation "joins" *j* to *S*, *S* invariably remains in place, whereas, if necessary, *j* is moved around.

> You may think of your bank account: even if interest is added or something paid into it, it remains *your* bank account.
>
> Actually, the Babylonian word for "interest" [on a loan] is "the joined".

“Heaping” may designate the addition of abstract numbers. Nothing therefore prevents from “heaping” (the number measuring) an area and (the number measuring) a length.

However, even “heaping” often concerns entities allowing a concrete operation.

The sum resulting from a “joining” operation has no particular name; indeed, the operation creates nothing new.

In a heaping process, on the other hand, where the two addends are absorbed into the sum, this sum has a name (*nakmartum*, derived from *kamārum*, “to heap”) which we may translate “the heap”.

There are also two subtractive operations.

One (*nasāḫum*/ZI), “from *B* to tear out *a*”, is the inverse of “joining”;
it is a concrete operation which presupposes *a* to be a constituent part of *B*.

The other is a comparison, which can be expressed “*A* over *B*, *d* goes beyond”.

- Even this is a concrete operation, used to compare magnitudes of which the smaller is not part of the larger.
- At times, stylistic and similar reasons call for the comparison being made the other way around, as an observation of *B* falling short of *A*.

The difference in the first subtraction is called “the remainder” (*šapiltum*, more literally “the diminished”).

In the second, the excess is referred to as the “going-beyond” (*watartum*/DIRIG).

“Multiplications”

Four distinct operations have traditionally been interpreted as multiplication.

First, there is the one which appears in the Old Babylonian version of the multiplication table. The Sumerian term (A.RÁ, derived from the Sumerian verb RÁ, “to go”) can be translated “steps of”.

For example, the table of the multiples of 6 runs:

1 step of 6 is 6
2 steps of 6 are 12
3 steps of 6 are 18
...

The second “multiplication” is defined by the verb “to raise” (*našûm*/ÍL/NIM).

The term appears to have been used first for the calculation of volumes: in order to determine the volume of a prism with a base of *G* SAR and a height of *h* KÙŠ, one “raises” the base with its standard thickness of 1 KÙŠ to the real height *h*.

Later, the term was adopted by analogy for all determinations of a concrete magnitude by multiplication. “Steps of” instead designates the multiplication of an abstract number by another abstract number.

The third “multiplication” (*šutakūlum*/GU$_7$.GU.$_7$), “to make *p* and *q* hold” (namely, hold a rectangle) – is no proper multiplication.

It always concerns two line segments *p* and *q*, and “to make *p* and *q* hold” means to construct a rectangle contained by the sides *p* and *q*.

Since *p* and *q* as well as the area *A* of the rectangle are all measurable, almost all texts give the numerical value of *A* immediately after prescribing the operation – “make 5 and 5 hold: 25” – without mentioning the numerical multiplication of 5 by 5 explicitly.

But there are texts that speak separately about the numerical multiplication, as “*p* steps of *q*”, after prescribing the construction, or which indicate that the process of “making hold” creates “a surface”.

The latter possibility is used in particular when the measures of one or both sides are unknown.

If a rectangle exists already, its area is determined by “raising”, just as the area of a triangle or a trapezium.

I shall designate the rectangle which is “held” by the segments p and q by the symbol $\sqsubset\!\sqsupset(p,q)$, while $\square(a)$ will stand for the square which a segment a “holds together with itself”

In both cases, the symbol designates the configuration as well the area it contains, in agreement with the ambiguity inherent in the concept of “surface”.

The corresponding numerical multiplications I shall write symbolically as $p{\times}q$ and $a{\times}a$.

The last “multiplication” (*eṣēpum*) is also no proper numerical multiplication.

“To repeat” or “to repeat until *n*” (where *n* is an integer small enough to be easily imagined, at most 9) stands for a “physical” doubling or *n*-doubling – for example that doubling of a right triangle with sides (containing the right angle) *a* and *b* which produces a rectangle ⊏⊐(*a*,*b*).

The problem “what should I raise to *d* in order to get *P*?” is a *division problem*, with answer $P \div d$.

Obviously, the Old Babylonian calculators knew such problems perfectly well.

a worker can dig *N* NINDAN irrigation canal in a day; how many workers will be needed for the digging of 30 NINDAN in 4 days?

> In this example the problem even occurs twice, the answer being $(30 \div 4) \div N$.

But division was no separate *operation* for them, only a problem type.

Of 1, its 2/3	40	16, its IGI	3 45	48, its IGI	1 15
Its half	30	18, its IGI	3 20	50, its IGI	1 12
3, its IGI	20	20, its IGI	3	54, its IGI	1 6 40
4, its IGI	15	24, its IGI	2 30	1, its IGI	1
5, its IGI	12	25, its IGI	2 24	1 4, its IGI	56 15
6, its IGI	10	27, its IGI	2 13 20	1 12, its IGI	50
8, its IGI	7 30	30, its IGI	2	1 15, its IGI	48
9, its IGI	6 40	32, its IGI	1 52 30	1 20, its IGI	45
10, its IGI	6	36, its IGI	1 40	1 21, its IGI	44 26 40
12, its IGI	5	40, its IGI	1 30		
15, its IGI	4	45, its IGI	1 20		

In order to divide 30 by 4, they first used a table where they could read (but they had probably learned it by heart in school) that IGI 4 is 15´;

afterwards they “raised” 15´ to 30 (even for that tables existed, learned by heart at school), finding 7°30´.

Finding (for instance) IGI 4 is spoken of as "detaching" it – probably referring to an original idea of detaching 1 out of4 parts.

Primarily, IGI *n* stands for *the reciprocal of n as listed in the table* or at least as easily found from it, not the number $\frac{1}{n}$ abstractly.

In this way, the Babylonians solved the problem $P \div d$ via a multiplication $P \cdot \frac{1}{d}$ *to the extent that this was possible*.

However, this was only possible if *n* appeared in the standard IGI table.

Firstly, that required that *n* was a "regular number", that is, that $\frac{1}{n}$ could be written as a finite "sexagesimal fraction".

However, of the infinitely many such numbers only a small selection found place in the standard table – around 30 in total.

In practical computation, that was generally enough. All technical constants – for example, the quantity of dirt a worker could dig out in a day – were indeed presupposed to be simple regular numbers.

The solution of “algebraic” problems, on the other hand, often leads to divisions by a non-regular divisor *d*.

> In such cases, the texts write “what shall I posit to *d* which gives me *A*?, giving immediately the answer “posit *Q*, *A* will it give you”.

That has a very natural explanation: these problems were constructed backwards, from known results. Divisors would therefore always divide, and the teacher who constructed a problem already knew the answer as well as the outcome of divisions leading to it.

To “posit to” means “write along with”, in the way multiplication exercices were written in more elementary teaching

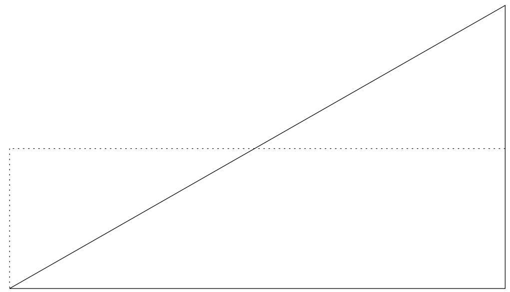
Halves

$\frac{1}{2}$ may be a fraction like any other: $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. This kind of half, if it is *the half of something* and not just a number, is found by raising that thing to 30´. Similarly, its $\frac{1}{3}$ is found by raising to 20´, etc.

But $\frac{1}{2}$ (in this case *necessarily* the half of something) may also be a "natural" or "necessary" half, that is, a half that could be nothing else.

The radius of a circle is thus the "natural" half of the diameter: no other part could have the same role.

Similarly, it is by necessity the exact half of the base that must be raised to the height of a triangle in order to give the area – as can be seen on the figure used to prove the formula.



This “natural” half had a particular name (*bāmtum*), which I shall translate “moiety”. The operation that produced it was expressed by the verb “to break” (*ḫepûm*/GAZ) – that is, to bisect, to break in two equal parts.

This meaning of the word belongs specifically to the mathematical vocabulary; in general usage the word means to crush or break in any way (etc.).

Square and “square root”

The product $a \cdot a$ played no particular role, neither when resulting from a “raising” or from an operation of “steps of”. A square, in order to be something special, had to be a geometric square.

But the geometric square did have a particular status. One might certainly “make *a* and *a* hold” or “make *a* together with itself hold”;

but one might also “make *a* confront itself” (*šutamḫurum*, from *maḫārum* “to accept/receive/approach/welcome”).

The square seen as a geometric configuration was a “confrontation” (*mitḫartum*, from the same verb).

Numerically, its value was identified with the length of the side. A Babylonian “confrontation” thus *is* its side while it *has* an area; inversely, our square *is* an area and *has* a side.

> Not understanding this, modern readers have sometimes claimed that the Babybolians did not distinguish a square from a square root.

When the value of a “confrontation” (understood thus as its side) is found, another side which it meets in a corner may be spoken of as its “counterpart” – *meḫrum* (also from *maḫārum*), used also for instance about the exact copy of a tablet.

In order to say that *s* is the side of a square area *Q*, a Sumerian phrase was used: “by *Q*, *s* is equal” – the Sumerian verb being ÍB.SI$_8$.

Sometimes, the word ÍB.SI$_8$ is used as a noun, in which case I shall translate it “the equal”.

In the arithmetical interpretation, “the equal” becomes the square root.

Just as there were tables of multiplication and of reciprocals, there were also tables of squares and of "equals".

They used the phrases "n steps of n, n^2" and "by n^2, n is equal" ($1 \leq n \leq 60$).

The resolution of "algebraic" problems often involves finding the "equals" of numbers which are not listed in the tables.

The Babylonians did possess a technique for finding approximate square roots of non-square numbers – but these *were* approximate.

The texts instead give the exact value, and once again they can do so because the authors had constructed the problem backward and therefore knew the solution.

Concerning the texts and translations that follow

The texts I present and explain in the following are written in Babylonian.

Basically they are formulated in syllabic (thus phonetic) writing.

All also make use of logograms that represent a whole word but do not indicate neither the grammatical form not the pronunciation (although grammatical complements are sometimes added to them).

With rare exceptions, these logograms are borrowed from Sumerian. Some correspond to technical expressions already used as such by the Sumerian scribes; IGI is an example. Others serve as abbreviations for Babylonian words,

> more or less like *viz* in English, which represents the shorthand for *videlicet* in medieval Latin manuscripts but is pronounced *namely*.

As said, our texts come from the second half of the Old Babylonian epoch, as can be seen from the handwriting and the language.

Unfortunately it is often impossible to say more, since almost all come from illegal diggings and have been bought by museums on the antiquity market in Baghdad or Europe.

We have no direct information about the authors of the texts. They never present themselves, and no other source speaks of them.

Since they knew to write they must have belonged to the category of scribes.

Since they knew to calculate, we may speak about them as “calculators”;

and since the format of the texts refers to a didactical situation, we may reasonably assume that they were school teachers.

All this, however, results from indirect arguments.

Plausibly, the majority of scribes never produced mathematics on their own beyond simple computation;

few were probably ever trained at the high mathematical level presented by our texts.

It is even likely that only a minority of school teachers *taught* such matters.

We cannot even be sure that the authors were teachers.

The format of the texts is certainly a school format, as we shall see: A teacher sets out a situation or problem, and and instructor explains what “you” should do.

Similarly, this sculpture is in “door format” – but where is the wall?

My English translations do not distinguish between syllabically and logographically written words.

Apart from that, they are “conformal” – that is, they are faithful to the original

- in the structure of phrases
- as well as by using always distinct translations for words that are different in the original and the same translation for the same word every time it occurs unless it is used in clearly distinct functions.

In as far as possible the translations respect the non-technical meanings of the Babylonian words (for instance “breaking” instead of “bisecting”)

and also the relation between terms (thus “confront itself” and “confrontation” – while “counterpart” had to be chosen unrelated of the verbal root in order to respect the use of the same word for the copy of a tablet).

This is not to say that the Babylonians did not have a technical terminology but only their everyday language;

but it is important that the technical meaning of a word be learned from its uses within the Old Babylonian texts and not borrowed (with the risk of being badly borrowed, as has often happened) from our modern terminology.

In order to avoid completely illegible translations, the principle is not followed to extremes; articles, punctuation and indications of order of magnitude have been added.

Clay tablets have names, most often museum numbers. The small problem that was quoted is the first one on the tablet BM 13901 – that is, tablet #13901 in the British Museum tablet collection.

Other names begin AO (Ancient Orient, Louvre, Paris), VAT (Vorderasiatische Texte, Berlin) or YBC (Yale Babylonian texts). TMS refers to the edition *Textes mathématiques de Suse* of a Louvre collection of tablets from Susa.

The tablets are mostly inscribed on both surfaces (“obverse” and “reverse”), sometimes in several columns, sometimes also on the edge; the texts are divided in lines read from left to right.

Techniques for the first degree

We shall mainly look at the Old Babylonian treatment of second-degree problems.

However, the solution of second-degree equations or equation systems often asks for first-degree manipulations. ,   ½

It will therefore be useful to start with a text which explains how first-degree equations are transformed and solved.

TMS XVI #1

1. The 4th of the width, from the length and the width I have torn out, 45´. You, 45´

2. to 4 raise, 3 you see. 3, what is that? 4 and 1 posit,

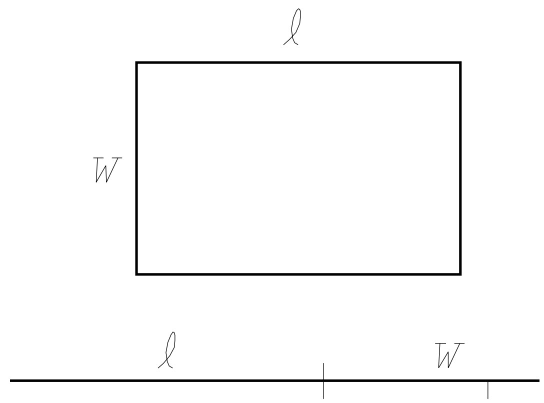
3. 50´ and 5´, to tear out, posit. 5´ to 4 raise, 1 width. 20´ to 4 raise,

4. 1°20´ you ⟨see⟩, 4 widths. 30´ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20´, 1 width, to tear out,

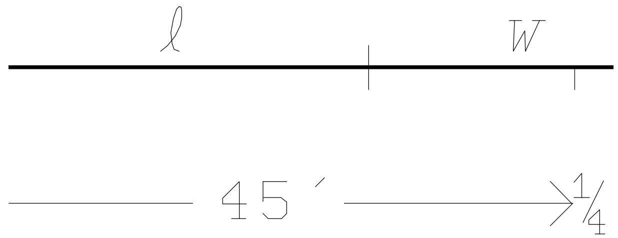
5. from 1°20´, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, heap, 3 you see.

Line 1 formulates an "equation" involving a fraction, lines 1-2 multiply it by the denominator and asks for the meaning of the outcome.

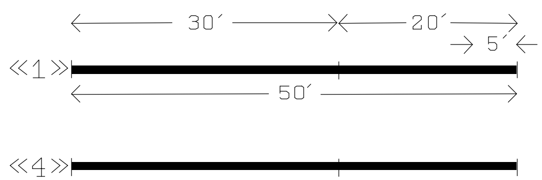
Lines 3–5 trace what happens to the single members. As we see, the "unknowns" are supposed to be 30´ and 20´.



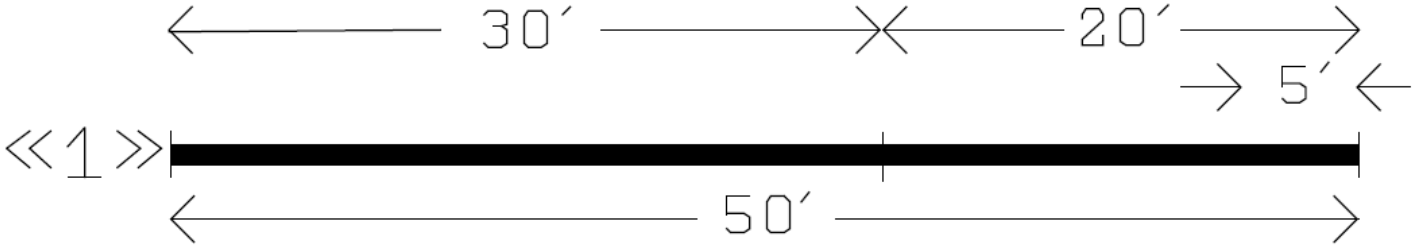
The geometry of TMS XVI #1



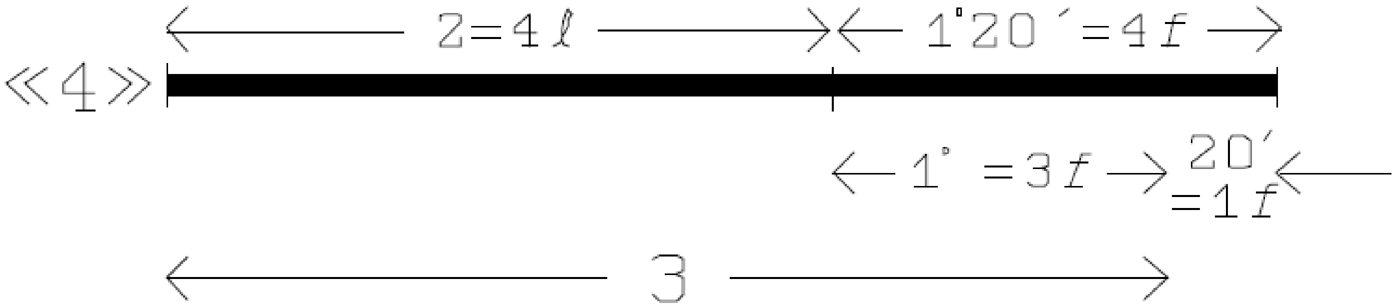
"The equation" of TMS XVI #1



Interpretation of TMS XVI, lines 3–5

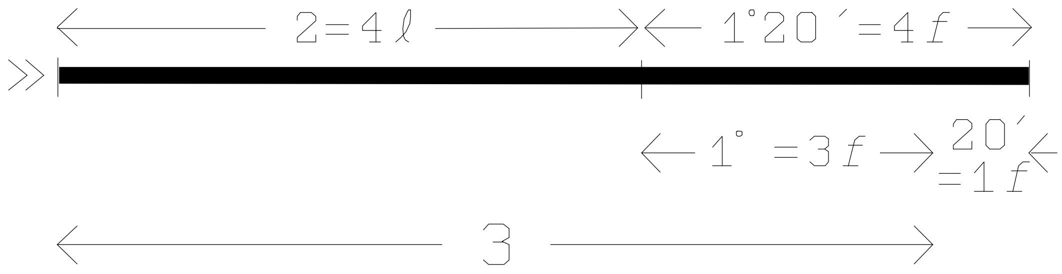

«1»
30′
20′
5′
50′


«4»


«4»
2=4ℓ
1°20′=4f
1° =3f
20′
=1f
3

6. IGI 4 detach, 15´ you see. 15´ to 2, lengths, raise, 30´ you ⟨see⟩, 30´ the length.

7. 15´ to 1 raise, 15´ the contribution of the width. 30´ and 15´ hold.



Interpretation of TMS XVI, line 5

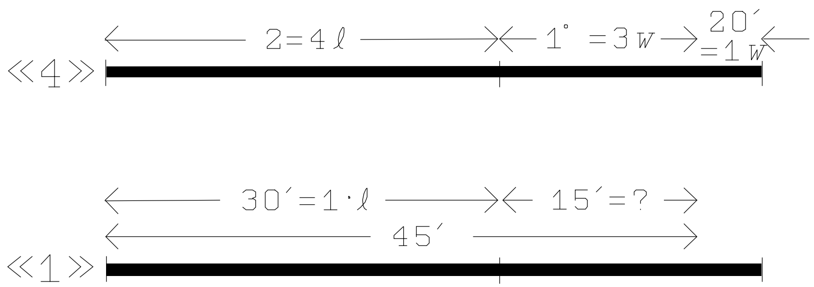
8. Since “The 4th of the width, to tear out”, it is said to you, from 4, 1 tear out, 3 you see.

9. IGI 4 de⟨tach⟩, 15´ you see, 15´ to 3 raise, 45´ you ⟨see⟩, 45´ as much as (there is) of widths.

10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1´ raise, 20´ you see.

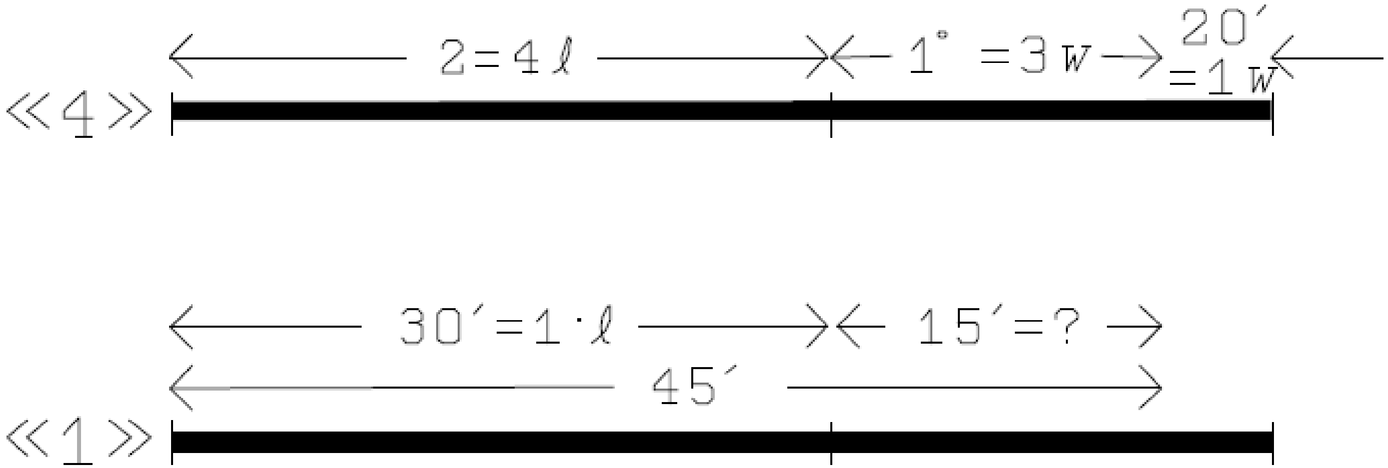
11. 20´ to 45´ raise, 15´ you see. 15´ from ^{30}15´ tear out,

12. 30´ you see, 30´the length.



Interpretation of TMS XVI, lines 6–12

Lines 6–12 reverses the multiplication and thus identifies the contributions of length and width.


2=4ℓ
1° =3w
20′
=1w
《4》
30′=1·ℓ
15′=?
45′
《1》

To observe:

1. The alternation of grammatical person. The teacher is 1st person singular presents the situations, an instructor explains what “you” should do, referring to the teacher as “he”.

2. The quotation from the statement in line 8.

3. The explicit concent of a coefficient “as much as (there is of)” *X* in line 9.

4. Width/“true width” in line 10. It *could* mean that the text distinguishes the “real” widths 30 and 40 NINDAN (180 respectively 120 metres) from the representing “school-yard” length and width 30´ and 20´ (3 respectively 2 metres).

The fundamental techniques for the second degree

BM 13901 #1

Obv. I

1. The surface and my confrontation I have heaped: 45´ is it. 1, the projection,

2. you posit. The moiety of 1 you break, 30´ and 30´ you make hold.

3. 15´ to 45´ you join: by 1, 1 is equal. 30´ which you have made hold

4. from the inside of 1 you tear out: 30´ the confrontation.

The problem can thus be expressed in symbols in this way:

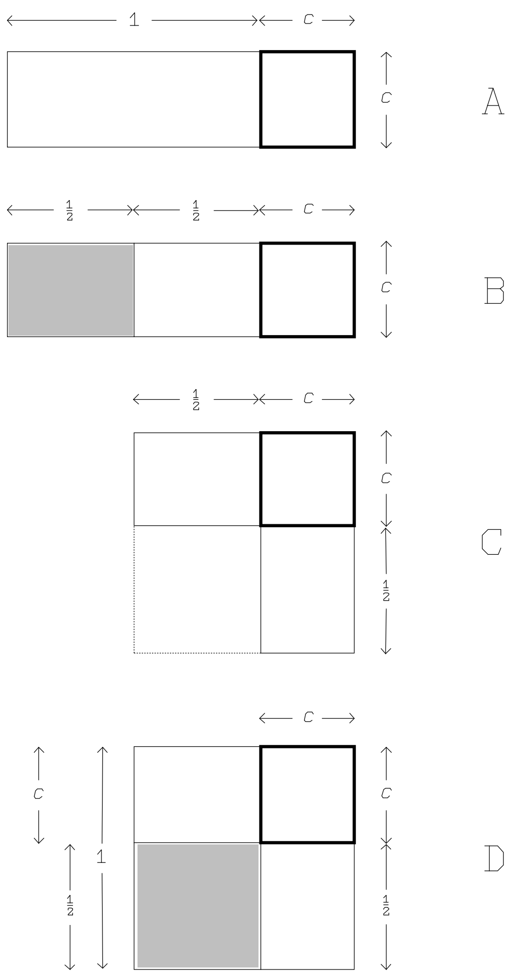
$$\square(c)+c = 45' \ (= \tfrac{3}{4}) \ .$$

It is solved by a "cut-and-paste" procedure.

For this purpose, the side ("confrontation") is provided with a breadth, a "projection" of 1.

That changes it into a rectangle, that can be bisected ("broken"), the outer part moved around so as to form with what remains in place a gnomon.

This allows a quadratic completion. From the side of the completed square the part that was moved is "torn out". That leaves the "confrontation".



The procedure of BM 13901 #1, in slightly distorted proportions

Those who are familiar with Euclid’s *Elements* will recognize a “naive” version of book II proposition 6.

BM 13901 #2

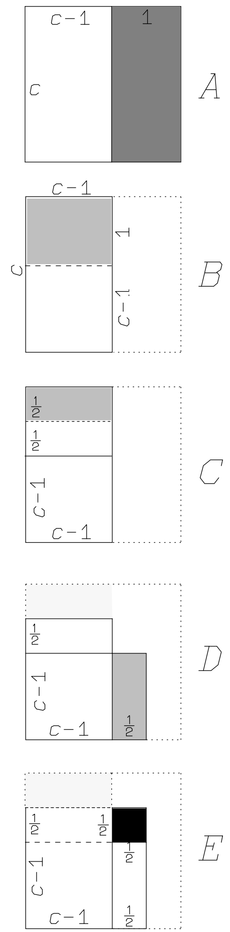
Obv. I

5. My confrontation inside the surface I have torn out: 14`30 is it. 1, the projection,

6. you posit. The moiety of 1 you break, 30´ and 30´ you make hold,

7. 15´ to 14`30 you join: by 14`30°15´, 29°30´ is equal.

8. 30´ which you have made hold to 29°30´ you join: 30 the confrontation.



The procedure of BM 13901 #2

In modern symbols, the problem corresponds to

$$\square(c)-c = 14`30\ .$$

The geometric procedure is the same as before: From the total surface (the square on the confrontation c) one confrontation has to be removed. It is therefore provided with a “projection” 1 and thus transformed into a rectangle.

Removal of this rectangle leaves another rectangle with sides c and c–1.

The is the same situation as we had in the previous problem, and the procedure is the same:

The excess of c over c–1 is bisected, the outer part moved around, and a quadratic completion is performed. Etc.

We observe that this time the confrontation is 30, not 30´.

If c = 30´, is would indeed exceed the suface.

YBC 6967

Obv.

1. The *igibûm* over the *igûm*, 7 it goes beyond

2. *igûm* and *igibûm* what?

3. You, 7 which the *igibûm*

4. over the *igûm* goes beyond

5. to two break: 3°30´;

6. 3°30´ together with 3°30´

7. make hold: 12°15´.

8. To 12°15´ which comes up for you

9. 1` the surface join: 1`12°15´.

10. The equal of 1`12°15´ what? 8°30´.

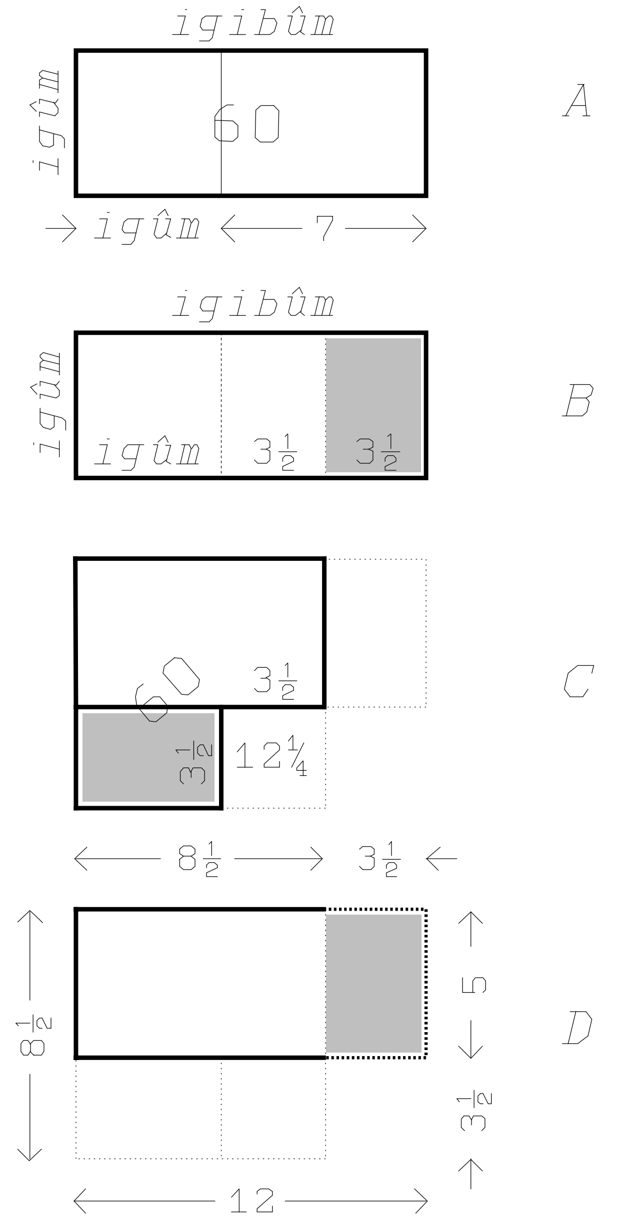
11. 8°30´ and 8°30´, its counterpart, lay down.

Rev.

1. 3°30´, the made-hold,

2. from one tear out,

3. to one join.



The procedure of YBC 6967

4. The first is 12, the second is 5.

5. 12 is the *igibûm*, 5 is the *igûm*.

This problem deals with a pair of numbers from the IGI-table, *igûm* and *igibûm* (the IGI and its IGI), whose product is supposed to be 60 and not 1.

This reflects how the IGI table was originally understood.

The numbers are represented by the sides of a rectangle with area 60.

The geometric procedure is the same.

10 minutes break

BM 13901 #10

Obv. II

11. The surfaces of my two confrontations I have heaped: 21°15´.

12. Confrontation (compared) to confrontation, the seventh it has become smaller.

13. 7 and 6 you inscribe. 7 and 7 you make hold, 49.

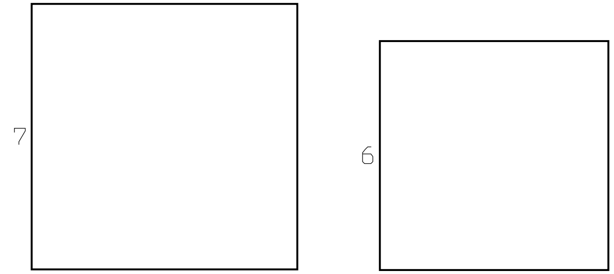
14. 6 and 6 you make hold, 36 and 49 you heap:

15. 1`25. IGI 1`25 is not detached. What to 1`25

16. may I posit which 21°15´ gives me? By 15´, 30´ is equal.

17. 30´ to 7 you raise: 3°30´ the first confrontation.

18. 30´ to 6 you raise: 3 the second confrontation.



The two squares of BM 13901 #10

In symbols, if the two sides are designated respectively c_1 and c_2:

$$\square(c_1)+\square(c_2) = 21°15' \;,\quad c_2 = c_1 - \frac{1}{7}c_1 \;.$$

This homogeneous problem is solved by a single false position. If c_1 is 7, then c_2 is 6, and the total area is 1`25; but is should be 21°50; etc.

Obv. II

44. The surfaces of my two confrontations I have heaped: 25´25´´.

45. The confrontation, two-thirds of the confrontation and 5´, NINDAN.

46. 1 and 40´ and 5´ over-going 40´ you inscribe.

47. 5´ and 5´ you make hold, 25´´ inside 25´25´´ you tear out:

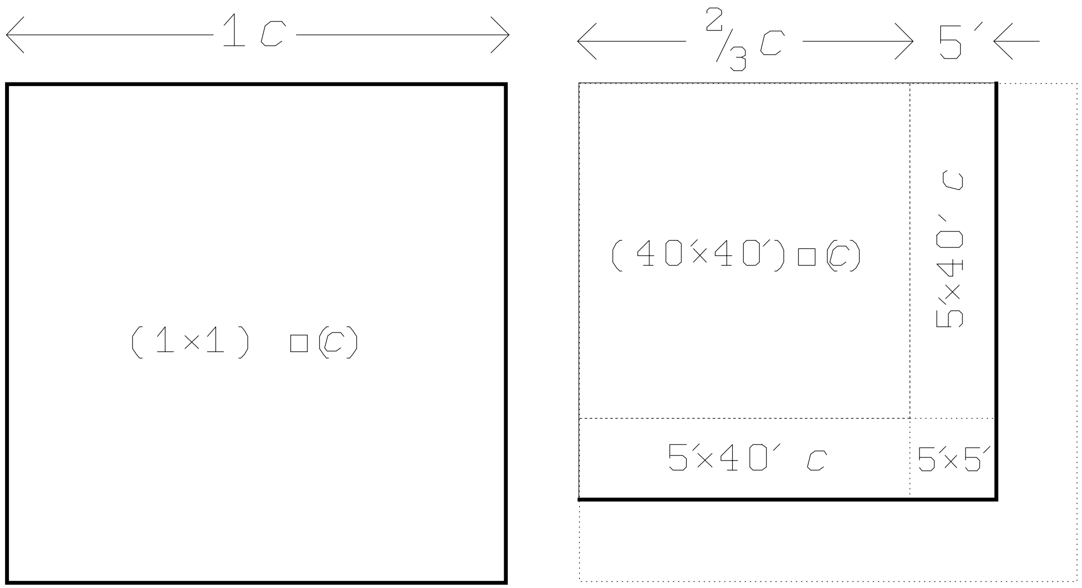
Rev. I

1. 25´ you inscribe. 1 and 1 you make hold, 1. 40´ and 40´ you make hold,

2. 26´40´´ to 1 you join: 1°26´40´´ to 25´ you raise:

3. 36´6´´40´´´ you inscribe. 5´ to 40´ you raise: 3´20´´

4. and 3´20´´ you make hold, 11´´6´´´40´´´´ to 36´6´´40´´´ you join:



The two squares of BM 13901 #14

5. by 36´17´´46´´´40´´´´, 46´40´´ is equal. 3´20´´ which you have made hold

6. inside 46´40´´ you tear out: 43´20´´ you inscribe.

7. IGI 1°26´40´´ is not detached. What to 1°26´40´´

8. may I posit which 43´20´´ gives me? 30´ its *bandûm*.

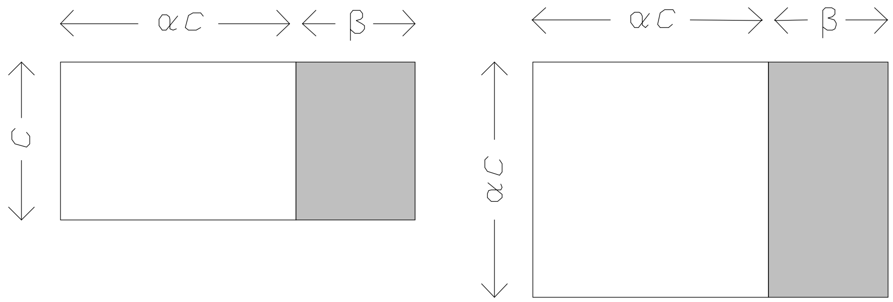
9. 30´ to 1 you raise: 30´ the first confrontation.

10. 30´ to 40´ you raise: 20´, and 5´ you join:

11. 25´ the second confrontation.

In symbols:

$$\square(c_1)+\square(c_2) = 25'25'' , \ c_2 = 40'\cdot c_1+5'.$$



Transformation of the problem $\alpha\square(c)+\beta c = \Sigma$

False positions do not work for inhomogeneous problems, so both sides are expressed in terms of a *new magnitude*, which we may call c:

$$c_1 = 1\cdot c \ , \ \ c_2 = 40'\cdot c+5' \ .$$

All in all:

$$1°26'40'' \square(c) + 2\cdot 3'20'' c = 25' \ .$$

This non-normalized problem is normalized by means of a multiplication by $1°26'40''$, which gives $\square(126'40''c)+2\cdot 3'20''\cdot(1°26'40''c) = 1°26'40''\cdot 25'$, which returns us to a situation we know.

Remarkably the doubling $2\cdot 3'20''\cdot$ is omitted because the writer knowns he will have to break the outcome.

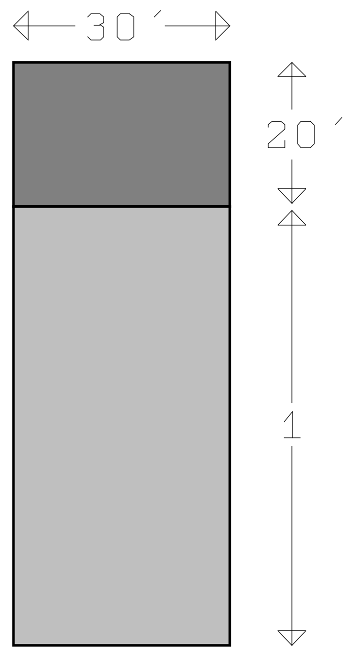
With one exception, the previous second-degree problems were from a single tablet probably written in the Mesopotamian south in the 18th century BCE.

Now we turn to a text written in Susa, thus in the periphery, probably toward 1600 BCE.

This origin may be the reason that it makes matters explicit which in the centre could be taken for granted and/or left to oral instruction

 similarly to the text TMS XVI, from which the first-degree problem was taken.

#1



TMS IX, #1

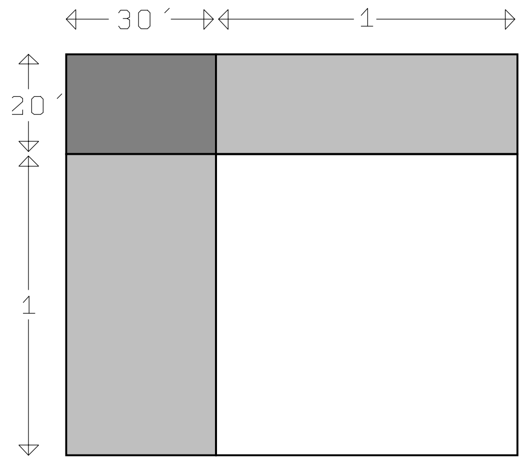
1. The surface and 1 length I have heaped, 40´. ¿30, the length,? 20´ the width.
2. As 1 length to 10´ the surface, has been joined,
3. or 1 (as) base to 20´, the width, has been joined,
4. or 1°20´ ¿is posited? to the width which 40´ together with the length ¿holds?
5. or 1°20´ toge⟨ther⟩ with 30´ the length holds, 40´ (is) its name.
6. Since so, to 20´ the width, which is said to you,
7. 1 is joined: 1°20´ you see. Out from here
8. you ask. 40´ the surface, 1°20´ the width, the length what?
9. 30´ the length. Thus the procedure.

This is a parallel to our symbolic transformation

$$\ell{\cdot}w+\ell = \ell{\cdot}w+\ell{\cdot}1 = \ell{\cdot}(w+1)\ ,$$

and shows that it works. Once more we see that the explanation is made on a figure with known dimensions, length 30´, width 20´, area 10´.

#2



TMS IX, #2

10. Surface, length, and width I have heaped, 1. By the Akkadian (method).

11. 1 to the length join. 1 to the width join. Since 1 to the length is joined,

12. 1 to the width is joined, 1 and 1 make hold, 1 you see.

13. 1 to the heap of length, width and surface join, 2 you see.

14. To 20´ the width, 1 join, 1°20´. To 30´ the length, 1 join, 1°30´.

15. ¿Since? a surface, that of 1°20´ the width, that of 1°30´ the length,

16. ¿the length together with? the width, are made hold, what is its name?

17. 2 the surface.

18. Thus the Akkadian (method).

Here, the equation to be transformed is

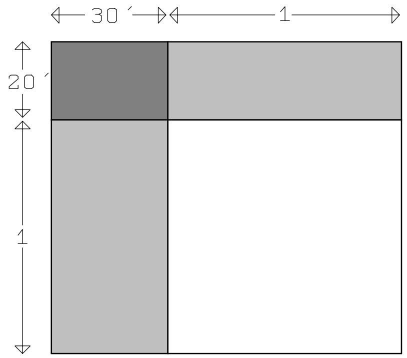
$$\ell{\cdot}w+\ell+w = \ell{\cdot}w+\ell{\cdot}1+w{\cdot}1$$

whence

$$\ell{\cdot}w+\ell+w+1 = \sqsubset\!\sqsupset(\ell+1,\mathrm{w}+1)$$

This is a variant of the quadratic completion. As we see, it has a name, “the Akkadian method”, which probably also covers its normal use as completion of a square gnomon.

The third problem on the same tablet shows how what has been taught here and in TMX XVI can be applied to a difficult problem. I insert symbolic translations and shall not go through the detailed calculations.



TMS IX, #2

TMS IX #3

#3

19. Surface, length, and width I have heaped, 1 the surface. 3 lengths, 4 widths heaped,

20. its 17th to the width joined, 30´.

$$\frac{1}{17}(3\ell+4w)+w = 30´ \ , \quad \sqsubset\!\sqsupset(\ell,w)+\ell+w = 1 \ .$$

21. You, 30´ to 17 go: 8°30´ you see.

22. To 17 widths 4 widths join, 21 you see.

23. 21 as much as of widths posit. 3, of three lengths,

24. 3, as much as lengths posit. 8°30´, what is its name?

25. 3 lengths and 21 widths heaped.

26. 8°30´ you see

27. 3 lengths and 21 widths heaped.

$$3\ell+(4+17)w = 3\ell+21w = 17\cdot30' = 8°30' \,.$$

28. Since 1 to the length is joined and 1 to the width is joined, make hold:

29. 1 to the heap of surface, length, and width join, 2 you see,

30. 2 the surface. Since the length and the width of 2 the surface,

31. 1°30´, the length, together with 1°20´, the width, are made hold,

32. 1 the joined of the length and 1 the joined of the width,

33. make hold, ¿1 you see.? 1 and 1, the various (things), heap, 2 you see.

34. 3 ..., 21 ..., and 8°30´ heap, 32°30´ you see;

35. so you ask.

$$\lambda = \ell+1 \;,\;\; \phi = w+1$$

$$\sqsubset\!\sqsupset(\lambda,\phi) = 2, \;\; 3\lambda+21\phi = 3\cdot(\ell+1)+21\cdot(w+1)$$

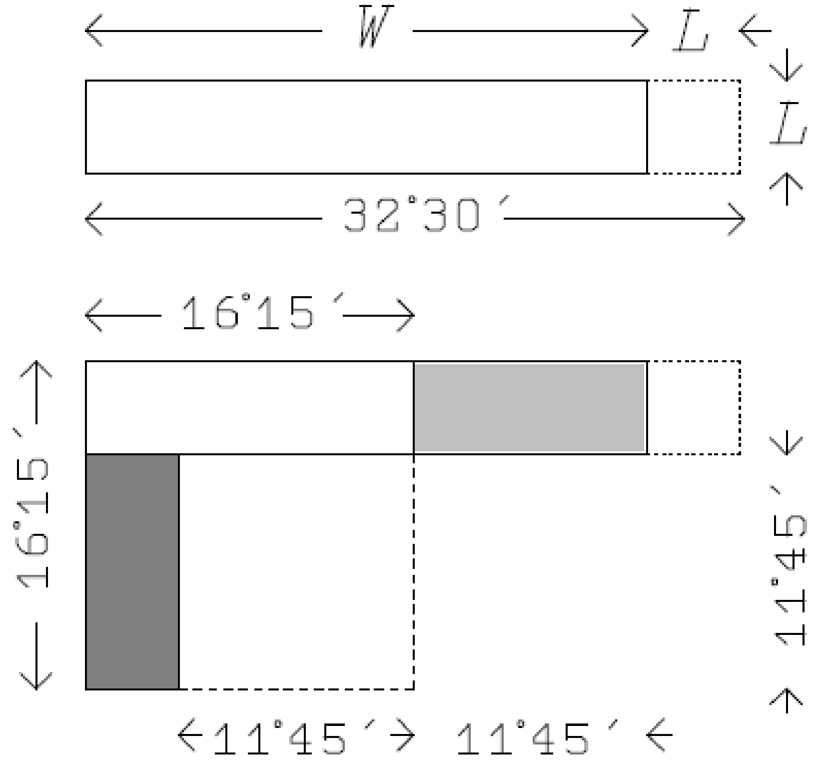
36. ... of widths, to 21, that heap:

37. ... to 3, lengths, raise,

Introducing $L = 3\lambda$, $W = 21\phi$

(no specific names in the text!)
and summing up we thus have

$L+W = 32°30´$, $\sqsubset\!\sqsupset(L,W) = 2`6°$.



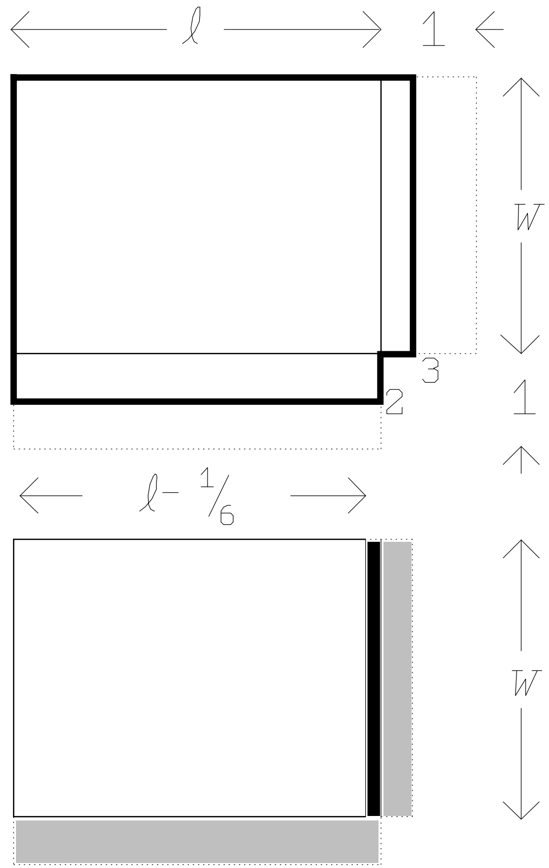
The cut-and-paste method of
TMS IX #3

This corresponds to *Elements* II.5, and is solved by a different cut-and-past procedure.

38. 1`3 you see. 1`3 to 2, the surface, raise:
39. 2`6 you see, ¿2`6 the surface?. 32°30´ the heap break, 16°15´ you ⟨see⟩.
40. {...}. 16°15´ the counterpart posit, make hold,
41. 4`24°3´45´´ you see. 2`6 ¿erasure?
42. from 4`24°3´45´´ tear out, 2`18°3´45´´ you see.
43. What is equal? 11°45´ is equal, 11°45´ to 16°15´ join,
44. 28 you see. From the 2nd tear out, 4°30´ you see.
45. IGI 3, of the lengths, detach, 20´ you see. 20´ to 4°30´
46. {...} raise: 1°30´ you see,
47. 1°30´ the length of 2 the surface. What to 21, the widths, may I posit
48. which 28 gives me? 1°20´ posit, 1°20´ the width
49. of 2 the surface. Turn back. 1 from 1°30´ tear out,
50. 30´ you see. 1 from 1°20´ tear out,
51. 20´ you see.

The next text seems to be from Larsa and to have been written some 15 years after this city was conquered by Hammurabi, and to reflect the beginning of sophisticated mathematics in the former Sumerian South.

As often at the beginning of a tradition, its terminology and structure is still vacillating. The beginning seems to tell a story.



The reduction of AO 8862 #2

I

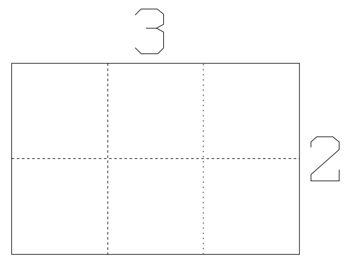
30. Length, width. Length and width
31. I have made hold: A surface I have built.
32. I turned around (it). The half of the length
33. and the third of the width
34. to the inside of my surface
35. I have joined: 15.
36. I turned back. Length and width
37. I have heaped: 7.

$$\sqsubset\!\sqsupset(\ell, w) + \frac{1}{2}\ell + \frac{1}{3}w = 15\ , \quad \ell + w = 7\ .$$

II

1. Length and width what?
2. You, by your proceeding,
3. 2 (as) inscription of the half
4. and 3 (as) inscription
5. of the third you inscribe:
6. IGI 2, 30´, you detach:
7. 30´ steps of 7, 3°30´; to 7,
8. the things heaped, length and width,
9. I bring:
10. 3°30´ from 15, my things heaped,
11. cut off:
12. 11°30´ the remainder.

Now follows a geometric determination of $\frac{1}{2}-\frac{1}{3}$



13. Do not go beyond. 2 and 3 make hold:
14. 3 steps of 2, 6.
15. IGI 6, 10´ it gives you.

16. 10´ from 7, your things heaped,

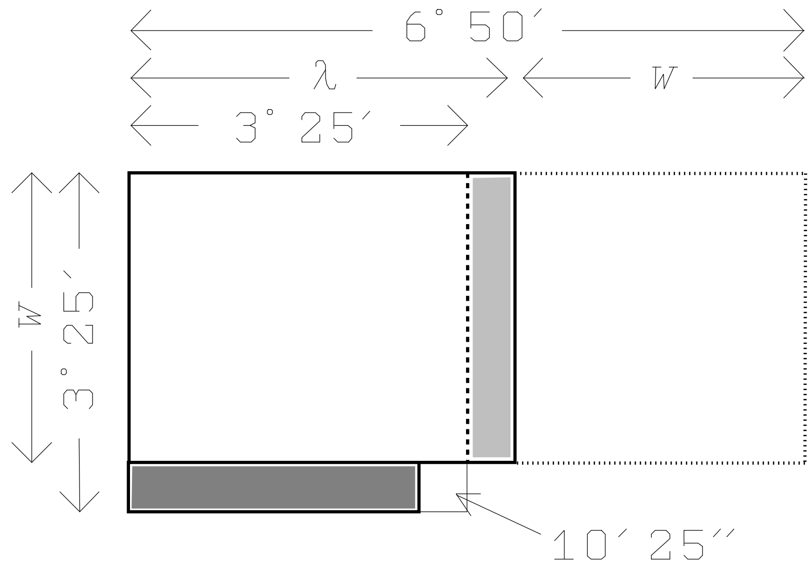
17. length and width, I tear out:

18. 6°50´ the remainder.

$\lambda + w$ = 7–10´ = 6°50´ , ⊏⊐(λ,w) = 11°30´

($\lambda = \ell$–10´).

Thus the same geometric situation as in TMS III #3



19. Its moiety, that of 6°50´, I break:
20. 3°25´ it gives you.
21. 3°25´ until twice
22. you inscribe; 3°25´ steps of 3°25´,
23. 11°40´25´´; from the inside
24. 11°30´ I tear out:
25. 10´25´´ the remainder. ⟨By 10´25´´, 25´ is equal⟩.
26. To the first 3°25´
27. 25´ you join: 3°50´,
28. and (that) which from the things heaped of
29. length and width I have torn out
30. to 3°50´ you join:
31. 4 the length. From the second 3°25´
32. 25´ I tear out: 3 the width.
32a. 7 the things heaped.
32b. 4, the length — 12, the surface
3, the width

As we see, this *could* have been solved in the same way as TMS IX no. 3, filling out the missing corner.

However, the author (and Old Babylonian mathematics in general) did not aim at training fixed algoriths but procedures that could be varied according to circumstances.

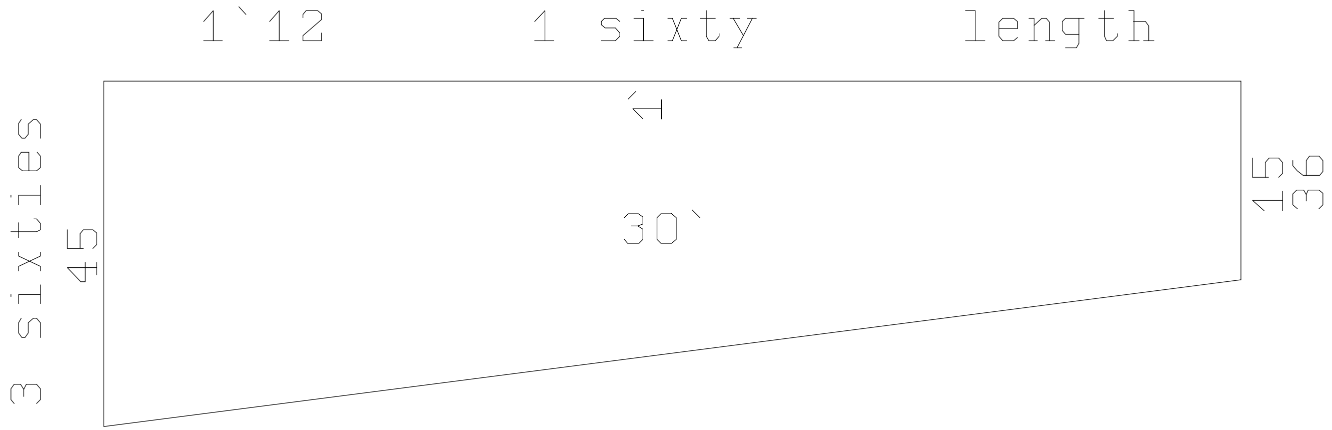
In contrast to classical Chinese mahematics, where use *and creation* of algorithms was central, Old Babylonian mathematics was not algorithmic.

Now we shall see how the “algebraic” technique was applied to what superficially looks as practical problems, but which actually are supra-utilitatian and highly artificial.

The first problem we shall look at appears to have been much appreciated – with minor variations it recurs over several text groups.

It looks as if it had to do with the mensuration of a field.

Since measurements are made by means of a reed which breaks twice, the first loss being given relatively and the second in part absolutely, the problem is of the second degree – which would never occur in gennuine surveying practice.



Obv.

1. A trapezium. I have cut off a reed. I have taken the reed, by its integrity

2. 1 sixty (along) the length I have gone. The 6th part

3. broke off for me: 1`12 to the length I have made follow.

4. I turned back. The 3rd part and $\frac{1}{3}$ KÙŠ broke off for me:

5. 3 sixty (along) the upper width I have gone.

6. With that which broke off for me I enlarged it:

7. 36 (along) the width I went. 1 BÙR the surface. The head (initial magnitude) of the reed what?

As a first step in the procedure, a single false position is introduced – namely that the unknown length of the reed is 1.

8. You, by your proceeding, (for) the reed which you do not know,

9. 1 may you posit. Its 6th part make break off, 50´ you leave.

10. IGI 50´ detach, 1°12´ to 1 sixty raise:

11. 1\`12 to ⟨1\`12⟩ join: 2\`24 the false length it gives you.

$$r = \frac{5}{6}R;\ 1`R = 1`12r$$

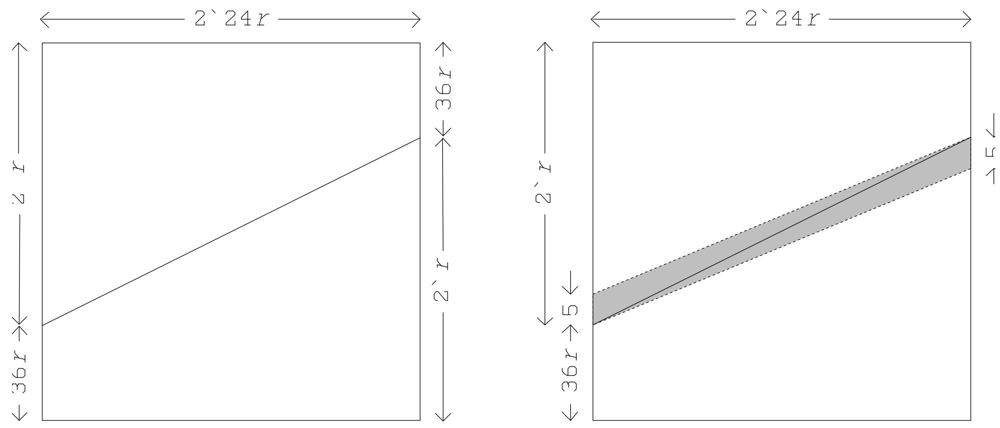
12. (For) the reed which you do not know, 1 may you posit. Its 3rd part make break off,

$$z = \frac{2}{3}r - \frac{1}{3}\text{KÙŠ}$$

13. 40´ to 3 sixty of the upper width raise:

14. 2\` it gives you. 2\` and 36 the lower width heap,

15. 2\`36 to 2\`24 the false length raise, 6\`\`14\`24 the false surface.



now follows the usual normalization be multiplication, with inherent change of variable

16. The surface to 2 repeat, 1`` to 6``14`24 raise
17. 6````14```24`` it gives you. And $\frac{1}{3}$ KÙŠ which broke off
18. to 3 sixty raise: 5 to 2`24, the false length,
19. raise: 12`. $\frac{1}{2}$ of 12` break, 6` make encounter,

Rev.

1. 36`` to 6````14```24`` join, 6````15``` it gives you.
2. By 6````15```, 2``30´ is equal. 6` which you have left
3. to 2``30`` join, 2``36` it gives you. IGI 6``14`24,
4. the false surface, I do not know. What to 6``14`24
5. may I posit which 2``36 gives me? 25´ posit.

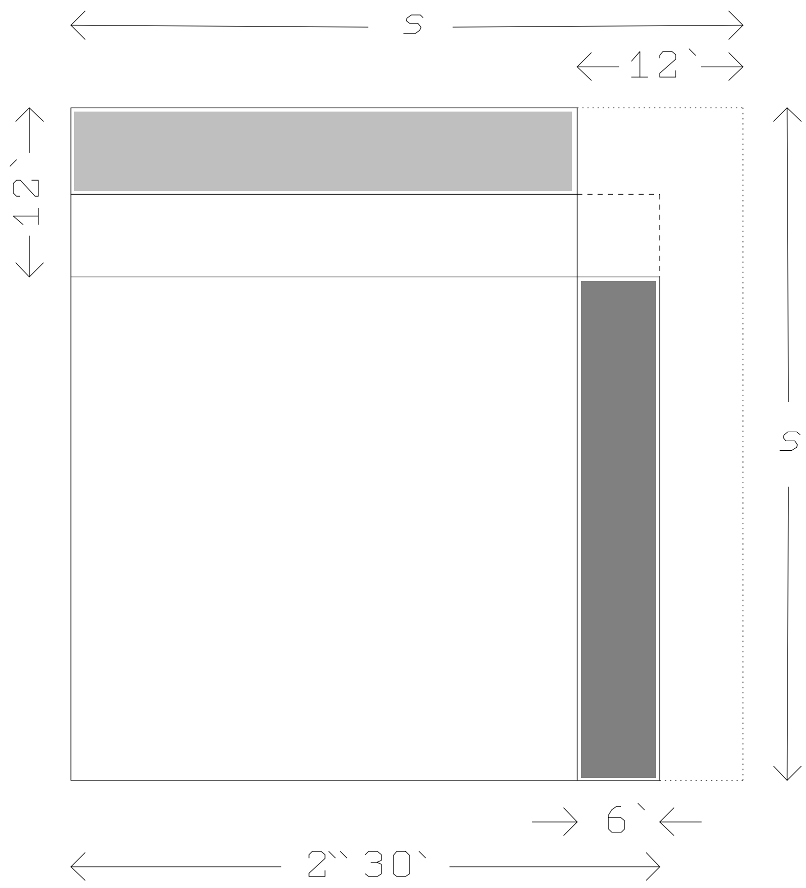
The lenght shortened by $\frac{1}{6}$ is thus 25´; the full length is found by another false position.

Now the change of variable is reversed by division

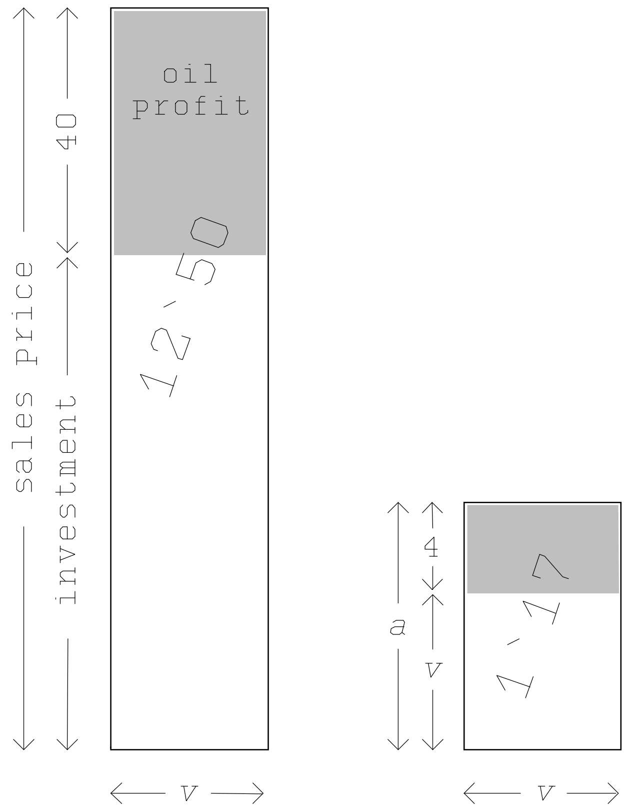
6. Since the 6th part broke off before,

7. 6 inscribe: 1 make go away, 5 you leave.

8. ⟨IGI 5 detach, 12´ to 25 raise, 5´ it gives you⟩. 5´ to 25´ join: $\frac{1}{2}$ NINDAN, the head of the reed it gives you.



TMS XIII



This text, once again from late Old Babylonian Susa, represents us with another distorted veriant of a real-world (here commercial) problem that leads to a second-degree problem.

(İ shall not go through the text – even well-trained mathematicians need considerable time to understand the method from the words, so it is not fit for a lecture; the diagram shows the central trick)

The problem deals with the purchase and sale of 12`50 SILÀ of oil. The difference between the rates (i.e., “how much oil for each shekel of silver”, inverse price) is also given (a–v = 4 shekel) together with the total profit ($^2/_3$ mina = 4 shekel). From this the two rates are to be found.

By means of a scaling in one dimension, the mathematical problem reduces to an analogue of the *igûm-igibûm* problem.

1. 2 GUR 2 PI 5 BÁN of oil I have bought. From the buying of 1 shekel of silver,
2. 4 SILÀ, each (shekel), of oil I have cut away.
3. $\frac{2}{3}$ mina of silver as profit I have seen. Corresponding to what
4. have I bought and corresponding to what have I sold?
5. You, 4 SILÀ of oil posit and 40, (of the order of the) mina, the profit posit.
6. IGI 40 detach, 1´30´´ you see, 1´30´´ to 4 raise, 6´ you see.
7. 6´ to 12`50, the oil, raise, 1`17 you see.
8. $\frac{1}{2}$ of 4 break, 2 you see, 2 make hold, 4 you see.
9. 4 to 1`17 join, 1`21 you see. What is equal? 9 is equal.

10. 9 the counterpart posit. $\frac{1}{2}$ of 4 which you have cut away break, 2 you see.

11. 2 to the 1st 9 join, 11 you see; from the 2nd tear out,

12. 7 you see. 11 SILÀ each (shekel) you have bought, 7 SILÀ you have sold.

13. Silver corresponding to what? What to 11 ¿SILÀ? may I posit

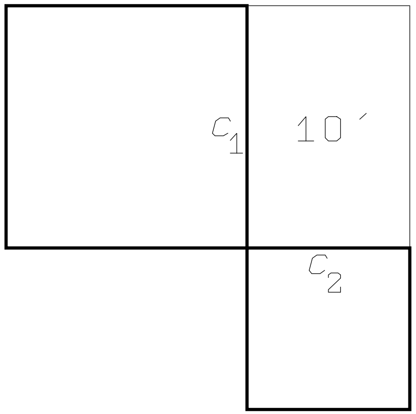
14. which 12`50 of oil gives me? 1`10 posit, 1 mina 10 shekel of silver.

15. By 7 SILÀ each (shekel) which you sell of oil,

16. that of 40 of silver corresponding to what? 40 to 7 raise,

17. 4`40 you see, 4`40 of oil.

BM 13901 #12



This problem comes from the tablet from which we got our first second-degree problems. Here, the sum of two square areas is given, together with the area of the rectangle contained by the two square-sides.

The method is to take the two square areas *as sides* of another rectangle, whose area is then the square on the area of the original rectangle. This leads to a rectangle-problem of *Elements*-II.5 type.

We observe that the calculator knows that $(c_1 \cdot c_2)^2 = c_1^2 \cdot c_2^2$.

That is, even though we seem to remain within the domain of geometry, we here have another instance of *representation*.

It is thereby probably the first demonstration of mathematics performing the famous “Indian rope trick”, where the fakir throws one end of a coiled rope into the air and then climbs the rope:

More elementary mathematics unfolding its potentiality to create higher mathematics.

Obv. II

27. The surfaces of my two confrontations I have heaped: 21´40´´.

28. My confrontations I have made hold: 10´.

29. The moiety of 21´40´´ you break: 10´50´´ and 10´50´´ you make hold,

30. 1´57´´21{+25}´´´40´´´´ is it. 10´ and 10´ you make hold, 1´40´´

$$\square(c_1)+\square(c_2) = 21'40'' \;,\quad ⊏⊐(\square(c_1),\square(c_2)) = 10'\times 10' = 1'40'' \;.$$

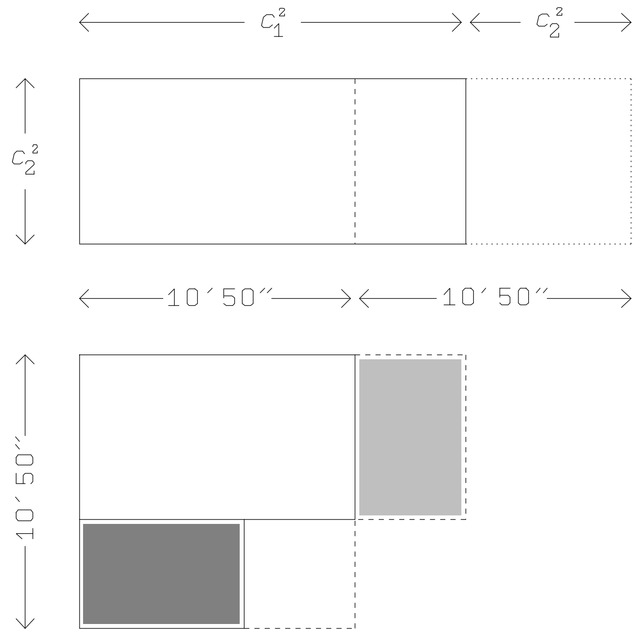
31. inside 1´57´´21{+25}´´´40´´´´ you tear out: by 17´´21 {+25}´´´40´´´´, 4´10´´ is equal.

32. 4´10´´ to one 10´50´´ you join: by 15´, 30´ is equal.

33. 30´ the first confrontation.

34. 4´10´´ inside the second 10´50´´ you tear out: by 6´40´´, 20´ is equal.

35. 20´ the second confrontation



In line 30 we observe that the outcome of $(10'50'')^2$ is given as

$$1'57''21\{+25\}'''40''''$$

Analysis of the calculation shows that a partial product has been inserted twice.

This shows that the calculation was made on a support where intermediate steps disappear from view once they are performed – that is, on something like a calculating board.

BM 13901 #23

This problem comes from the same tablet, and is remarkable in a different way.

Its terminology is highly unusual and points to lay Akkadian surveyors;

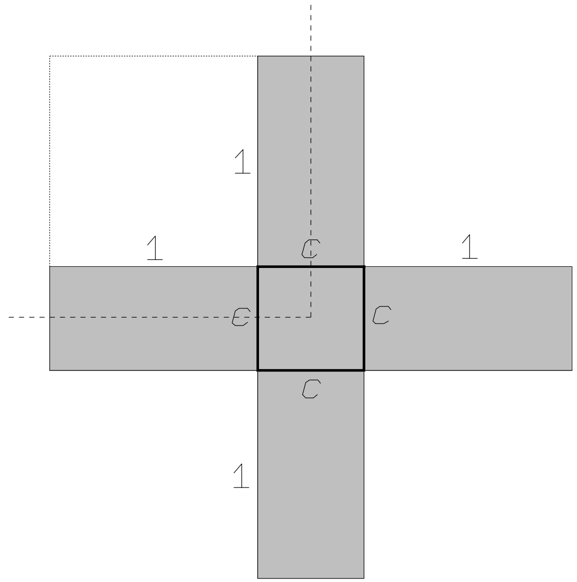
it is the only text in the complete record about a single square where the side is 10 (here, in a compromise with school habits, 10´).

It is indeed a “last problem before New-Year break”, and shows the application of “algebra” to an age-old riddle – one of a small group of riddles from which the scribe school created the “algebraic” technique.

The method used is also unusual; it was later to inspire al-Khwārizmī’s first geometric proof.

The problem, though now solved in the usual “*Elements* II.6” manner, is last found in the historical record in Luca Pacioli’s *Summa* from 1494 (CE) – still with solution 10.

Rev. II



11. About a surface, the four widths and the surface I have heaped, 41´40´´.

12. 4, the four widths, you inscribe. IGI 4 is 15´.

13. 15´ to 41´40´´ you raise: 10´25´´ you inscribe.

14. 1, the projection, you join: by 1°10´25´´, 1°5´ is equal.

15. 1, the projection, which you have joined, you tear out: 5´ to two

16. you repeat: 10´, NINDAN, confronts itself.

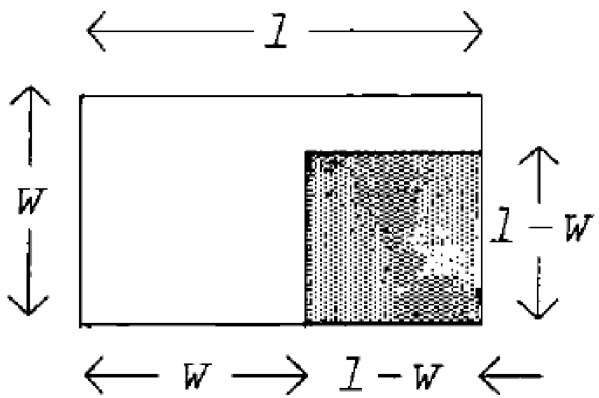
$$4c+\square(c) = 41'40''$$

YBC 6504 #4

Four problem types together form an elementary cluster: the area of a rectangle given together with either of its sides or with the sum of or the difference between the two sides.

The tablet YBC 6504 contains a variant: Instead of the area of the rectangle, the area of what remains after tearing out the square on the difference between the sides is given (a quasi-gnomon).

The fourth problem presents us with one of the rare mistakes (beyond wrong numbers) in the Babylonian record.



Rev.

11. So much as length over width goes beyond, made encounter, from inside the surface I have torn out:

12. 8´20´´. 20´ the width, its length what?

$$\sqsubset\!\sqsupset(\ell,w)-\square(\ell-w) = 8´20´´ \ , \quad w = 20´ \ .$$

13. 20´ made encounter: 6´40´´ you posit.

14. 6´40´´ to 8´20´´ you join: 15´ you posit.

15. By 15´, 30´ is equal. 30´, the length, you posit.

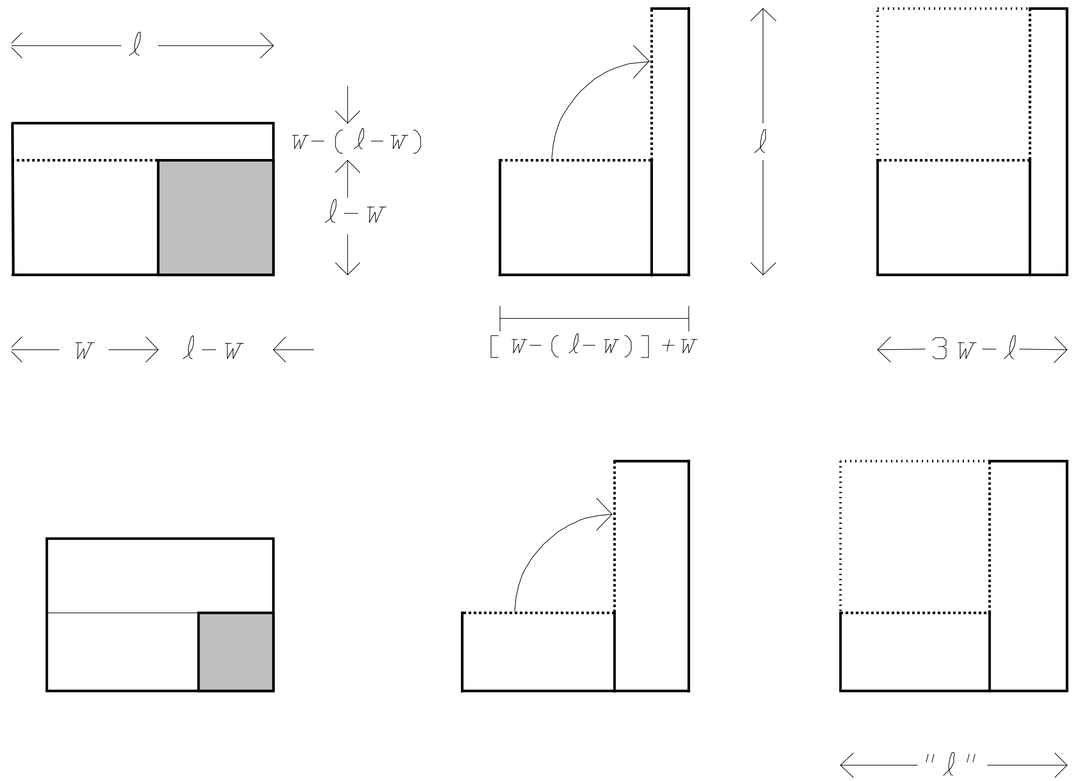
It is thus claimed that $\square(\ell) = \sqsubset\!\sqsupset(\ell,w)-\square(\ell-w)+\square(w)$

that is, $\ell = \sqrt{(3w-\ell)\cdot\ell}$

Geometrically, the mistake is easily understood:

The quasi-gnomon is cut and opened up. That gives another quasi-gnomon, where the square on the width is missing and can be filled out. What results, however, is not the square on ℓ – *unless*, as here, $\ell = \frac{3}{2}w$.

In geometric reasoning, great care should be taken, not to be mislead by what is “immediately seen”.



General characteristics

The techniques I have described until now were also applied to more intricate problems – indeterminate equations, a bi-biquadratic problem. A new technique (factorization) was used to serve irreducible cubic equations.

All this I shall leave aside, The technical mathematics I have presented already will certainly be ample for all but mathematicians. In the written material (the book *Algebra in Cuneiform*) those who want more can find all of it.

Who gets really hungry can ask for my extensive *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*.

Instead I shall take up two general questions:

To which extent do my drawings correspond to what the Babylonian calculators did?

and, in which sense is this “an algebra”?

On the tablets, drawings were only used as illustrations of the problem statement (for instance, in the “broken reed” problem.

Clay tablets are indeed not fit as basis for cut-and-paste procedures. Once a line is drawn, if is difficult to get rid of it – not to speak of moving it around.

Drawings must have been made on a different medium.

Sand strown on the paved floor of the school yard has been suggested for the first phase of the teaching of the script. If used for that purpose, it could also be used in the present case.

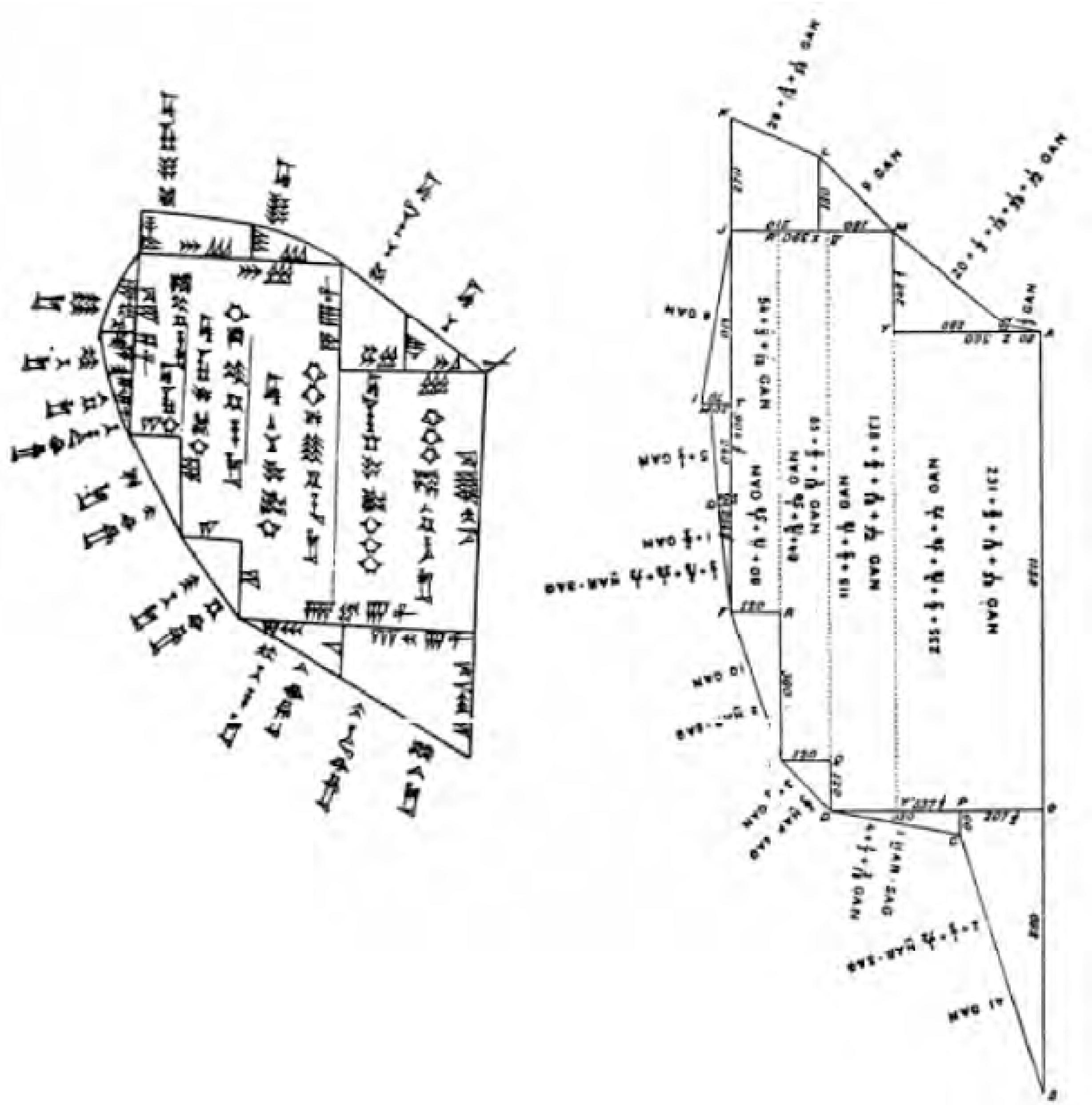
A dustboard is another possibility; it was used at least by the Phoenicians in the first millennium BCE.

My drawings were made according to modern ideals (Euclidean ideals, at least as these have been shaped by print culture.

A large number of field plans have survived from the Ur III epoch. They are different. They are structure diagrams.

They are not true to proportions, and only right angles (those that served area calculation) are rendered as such. Other angles – "wrong" angles – may be far from their true value.

Here is an example: Left as drawn
on the tablet, right as the true
proportions can be reconstructed from the numbers.



Many texts give us reasons to believe that they were satisfied with rudimentary structure diagrams.

Not strange. Who is familiarized with the Old Babylonian techniques will need nothing but a rough sketch in order to follow the reasoning;

there is not even any need to perform the divisions and displacements, the drawing of the rectangle alone allows one to grasp the procedure to be used.

As we may perform a mental computation, making at most notes for one or two intermediate results, we may also become familiar with “mental geometry”, at most assisted by a rough diagram.

Practising “mental geometry” presupposes that one has first trained concrete geometry; real drawings of some kind must thus have existed.

For reasons of convenience and in agreement with the majority of historians of mathematics, I have so far spoken of an Old Babylonian “algebra” without settling the meaning one should ascribe to this modern word in a Babylonian context.

Which are the similarities, which are the differences?

First of all: the modern algebra to which the Old Babylonian technique might perhaps be assimilated is *a technique*, namely the practice of equations.

Even if we disregard group theory and all its kin, modern algebra as it has developed since the 16th century may also be *theory* (concerning the link between coefficients and roots, etc.).

The Babylonians might use a term for “coefficient” (“as much as there is of”), but we have no reason the believe that they went beyond that.

Modern equations algebra is obviously based on equations. Which were their Old Babylonian counterparts?

They may indicate the total measure of a combination of magnitudes (often but not always geometric magnitudes);

they may declare that the measure of one combination equals that of another one;

or that the former exceeds or falls short of the latter by a specified amount.

The principle does not differ from than of any applied algebra, and thus not from the equations on which an engineer or an economist operate today. In this sense, the Old Babylonian problem statements are true equations.

But there is a difference.

Today's engineer *operates on* his equations: the magnitudes he moves from right to left, the coefficients he multiplies, the functions he integrates, etc. – all of these exist only as elements of the equation and have no other representation.

As a rule, the operations of the Babylonians, on the contrary, were realized within a *different* representation, that of measured geometric quantities.

Only first-degree transformations like those of TMS XVI #1 and TMS IX #3 constitute a partial exception;

TMS XVI #1 is indeed an explanation of how operations directly on the words of the equation are to be understood in terms of the geometric representation. Once that had been understood, TMS IX #3 could probably operate directly on the level of words.

But TMS XVI #1 is no problem solution, and in TMS IX #3 the first-degree transformation is subordinate to geometric operations.

Since Greek Antiquity, the solution of a mathematical problem is called “analytic” if it starts from the presupposition that the problem is *already* solved;

that allows us to examine – “to analyze” – the characteristics of the solution in order to understand how to construct it.

The antithesis of the “analytical” method is the “synthesis, in which the solution is constructed directly, after which this solution is shown to be indeed valid.

This is the proof style of Euclid’s *Elements*, and since Antiquity it is the consistent complaint that this makes it more difficult than necessary to understand the work:

One sees well that each step of a proof is correct, and therefore has to accept the end result as irrefutable; but one does not understand the reasons which make the author take the single step; the author appears shrewd rather than pedagogical.

A solution by equation is always analytical – “we suppose that the solution exists and call it *x*”. So, when Viète wanted to make algebra culturally legitimate he called it *analysis*.

Even the Old Babylonian cut-and-paste solutions are analytic. We presuppose that squares or rectangles etc. exist, we draw the total situation in which they participate. Then we manipulate these hypothetically existing entities in order to to see what can be said about them.

Most of the procedures of the Babylonian “algebraic” problems are “homomorphic” though not “isomorphic” analogues of ours, or at least easily explained in term of modern algebra.

These shared characteristics – statements shaped as equations, analysis, homomorphic procedures – have led many historians of mathematics to speak of a “Babylonian algebra”.

But there is a further reason for this characterization, a reason that may be more decisive.

Today’s equation algebra possesses a neutral “fundamental representation”: abstract numbers.

This neutral representation is an empty container that can receive all kinds of measurable quantities: distances, areas, electric charges and currents, population fertilities, etc.

Greek geometric analysis, on the other hand, concerns nothing but the geometric magnitudes it deals with, these represent nothing but what they are.

> Even when Apollonius applies *Elements* II where we would use algebra, the representation is of geometric entities by geometric entities.

In this respect, the Babylonian technique is closer to modern equation algebra than is Greek analysis.

Its line segments may represent areas, prices (better, inverse prices) – and in other texts numbers of workers and the number of days they work, and the like.

We might believe (because we are habituated to confound the abstract geometric plan and the paper on which we draw) that geometry is less neutral than abstract numbers – we know perfectly well to distinguish the abstract number 3 from 3 pebbles but tend to take a nicely drawn triangle for the triangle itself.

But even if we stay in our confusion we must admit that from the *functional point of view*, the Old Babylonian geometry of measured magnitudes is also an empty container.

Apart from replacing *numbers* with *measurable geometric quantities*, the Old Babylonian technique is thus rather similar to modern equation algebra.

If the modern technique is understood as an "algebra" in spite of its immense conceptual distance from group theory and its descendants, it seems reasonable to classify the Old Babylonian technique under the same heading.

That does not mean that there are no differences; there are, and even important differences; but not of a kind that would normally be used to separate "algebra" from what is not algebra.

Apart from the representation by a geometry of measurable magnitudes, the most important difference is probably that Old Babylonian second- (and higher-)degree algebra had no practical application.

No practical problem within the horizon of an Old Babylonian working scribe asked for the application of higher algebra.

All problems beyond the first degree are therefore artificial, and all are constructed backwards from a known solution (many first-degree problems are so, too).

For example, the author considers a square of side 10´ and then calculates that the sum of the four sides and the area is 41´40´´.

From this he then formulates the *problem* to go the opposite way, to find the side from this sum.

This kind of algebra is very familiar today. It allows teachers and textbook authors to construct problems for school students for which they may be sure of the existence of a reasonable solution.

The difference is that *our* artificial problems are supposed to train students in techniques that will later serve in “real-life” contexts. The Babylonian technique was supra-utilitarian.

The background

A different question is why this technique – algebraic or not – was created.

Why would anybody engage in developing, maintanining and transmitting this kind of knowledge? In order to understand that we must look at its social support

Old Babylonian mathematics was not the high-status diversion of wealthy and highly intelligent amateurs, as Greek mathematicians were or aspired to be.

According to the format of its texts it was taught in the scribe school – hardly to all students, not even among those who went through the full standard curriculum, but at least to a fraction of future scribes (or future scribe school masters only?).

But this is no more than a reasoned inference. The “doors” in the campus exhort us to take care.

The word “scribe” might mislead.

The scribe certainly knew to write. But the ability to calculate was just as important – originally, writing had been invented as subservient to accounting, and this subordinated function with respect to calculation remained very important.

The modern colleagues of the scribe are engineers, accountants and notaries.

Therefore, what was taught number- and quantity-wise in the scribe school should not be understood primarily as “mathematics” but rather as *calculation*.

The scribe was supposed to be able to *find the correct number*, be it in his engineering function, be it as an accountant.

Even problems that do not consider true practice always concern measurable magnitudes, and they always ask for a numerical answer (as we have seen).

It might be more appropriate to speak of the algebra as “pure calculation” than as (unapplied and hence) “pure” mathematics.

That is one of the reasons that many of the problems that have no genuine root in practice none the less speak of the measurement and division of fields, of the production of bricks, of the construction of siege ramps, of purchase and sale, and of loans carrying interest.

One may learn much about daily life in Babylonia (*as it presented itself to the eyes of a professional scribe*) through the *topics* spoken of in these problems, even when their mathematical substance is wholly artificial.

If we really want to find Old Babylonian “mathematicians” in an approximately modern sense, we must look to those who *created* the techniques and discovered how to *construct* problems that were difficult but could still be solved.

For example we may think of the problem TMS XIX #2 (which I have not presented but just referred to): to find the sides ℓ and w of a rectangle from its area and from the area of the rectangle $\sqsubset\!\sqsupset(d, ❒(\ell))$. This is a problem of the eighth degree.

Without systematic work of theoretical character, perhaps with a starting point similar to BM 13901 #12, it would have been impossible to guess that it was bi-biquadratic, and that it can be solved by means of a cascade of three successive quadratic equations.

But this kind of theoretical work has left no surviving written traces

One function and purpose of the "algebra" may have been to train the manipulation of difficult numbers.

Another one was ideological, the support for scribal pride. The ability to solve complicated problems would fit perfectly as an ingredient of scribal "humanism" as I discussed it in the previous lecture.

To find the area of a rectangular field from its length and width was not suited to induce much self-respect – any bungler in the trade could do that.

But to find a length and a width from their sum and the area they would “hold” was already more substantial;

to find them from data such as those of the nightmarish informations of the broken-reed problem – that would allow one to feel as a *real* scribe, as somebody who could command the respect of the non-initiates

> (although the non-initiates would hardly understand and therefore not discover).

Origin and heritage

One way to explain socio-cultural structures and circumstances argues from their function: if the scribe school for centuries expended much effort to teach advanced mathematics and even more on teaching Sumerian, then these activities must have had important functions

– if not as direct visible consequences then indirectly.

Another way to explain them – no alternative but rather the other side of the coin – is based on historical origin. Who had the idea, and when?

Or, if no instantaneous invention is in focus, how did the phenomenon develop, starting from which earlier structures and conditions?

Giving just a superficial outline of the complex arguments unravelling this story would ask for a full lecture, so I shall just sketch the outcome.

A sketch of the underlying argument will be found in *Algebra in Cuneiform*.

At some moment between 2200 and 1800 BCE, “lay”, that is, non-scribal Akkadian surveyors invented the trick of the quadratic completion (in one of our Susa texts called “the Akkadian method”).

That allowed them to solve this cluster of riddles:

$$c+\square(c) = 110$$

$$_4c+\square(c) = 140$$

$$\square(c)-c = 90$$

$$\square(c)-{_4c} = 60 \ (?)$$

$$\ell+w = \alpha\ , \quad \sqsubset\!\sqsupset(\ell,w) = \beta$$

$$\ell-w = \alpha\ , \quad \sqsubset\!\sqsupset(\ell,w) = \beta$$

$$\ell+w = \alpha\ , \quad (\ell-w)+\sqsubset\!\sqsupset(\ell,w) = \beta$$

$$\ell-w = \alpha\ , \quad (\ell+w)+\sqsubset\!\sqsupset(\ell,w) = \beta;$$

$$d = \alpha\ , \quad \sqsubset\!\sqsupset(\ell,w) = \beta\ .$$

$_4c$ here stands for “the 4 sides” and $\square(c)$ for the area of a square, d for the diagonal and $\sqsubset\!\sqsupset(\ell,w)$ for the area of a rectangle.

Beyond that, there were problems about two squares (sum of or difference between the sides given together with the sum of or difference between the areas);

a problem in which the sum of the perimeter, the diameter and the area of a circle is given,

and *possibly* the problem $d–c = 4$ concerning a square, with the pseudo-solution $c = 10$, $d = 14$;

two problems about a rectangle, already known before 2200 BCE, have as their data, one the area and the width, the other the area and the length.

One may guess that the trick was originally devised to make this group grow to four.

That seems to be all – but in an oral culture, the ability to solve a small set of riddles known to the insiders but not to outsiders is all that is needed for professional pride.

In a school environment, it does not suffice. That is illustrated by what happened to other riddles like the “hundred fowls” when they were adopted into a school: the number of variants increased.

> (“I bought 100 fowls for 100 coins. A goose cost 4, a hen 3, and chickens were 3 for a coin”.)

So, when the Old Babylonian school became “humanist”, the number of riddles was augmented,

– variations by means of coefficients was introduced (asking for a new trick),
– and the carrying capacity of the technique through representation was tested.

This produced “Old Babylonian algebra”.

Around 1600, the Old Babylonian social system collapsed, and the school disappeared. Other aspects of Old Babylonian scholarly culture survived within “scribal families”, but mathematics beyond the useful minimum was forgottent.

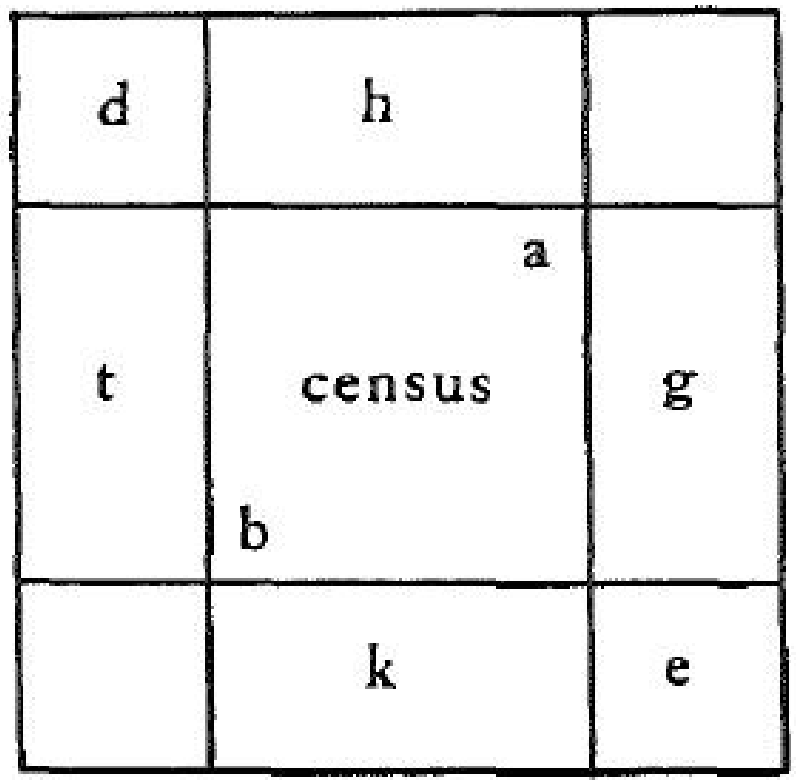
Around 500 and again around 300 BCE, scholar-scribes (now engaged in mathematical astronomy) tried to reintroduce sophisticated mathematics. They did that by adopting once more the old surveyor’s riddles, which were still around in the lay environment. Both episodes were ephemeral.

But the riddles were still around, and around 400 BCE, Greek geometers took up the basic riddles and submitted them to “critique”, asking *why* and *under which conditions* the technique worked. That resulted in *Elements* II.

> That is exactly Kant’s definition of critique: *Möglichkeit und Grenzen*, “possibility and limits”.

Around 820, al-Khwārizmī wrote a treatise in Baghdād about a wholly different kind of algebra (a purely numerical technique, where an amount of money and its square root represent other entities).

Wanting to provide the solutions with proofs in semi-Greek style, he borrowed the geometric riddle technique, first the trick of the “four sides and the area”.



When Leonardo Fibonacci wrote his *Liber abbaci* in 1202, he believed that these proofs were the essence of algebra. That, however, was without consequence, as I shall tell in a later lecture.